Sadagopan Rajesh TEST: RELATIONS and FUNCTIONS JULY 05, 2024

Maximum time: 30 *minutes*

KEM7 - Advanced Maths for Std 12 only @ *ABIMS*

Try on your own ! Don't use calculators ! Think and Answer !

Name:____ Standard:__________

1. $Y = \{LCM(x, 60) : x \in \mathbb{W}; 1 \leq x \leq 20\}$ Then $|Y|$ is _______.

2. A and B are *sets* such that $|A| = 20; |B| = 12$. Which of the following <u>could be</u> *true*?

   A. $|A \cup B| = 35$ B. $A \cap B = 8$ C. $|B - A| = 12$ D. $|A - B| = 7$

3. A *Relation* is given by $R : A \to B = \Big\{\big(a, b\big) : a \in A; b \in B; \big(a + b\big) \text{ is } prime\Big\}$

   Its *roster form* is given by $R = \Big\{\big(20, 3\big), \big(21, 2\big), \big(22, 1\big)\Big\}$.

   Which of the following is <u>definitely</u> *true*?

   A. *Domain* of R is $A = \{20, 21, 22\}$. B. *Co-Domain* of R is $B = \{1, 2, 3, 4\}$.

   C. *Domain* of R is $A = \{20, 21, 22, 23\}$. D. none of these

4. $F : \dot{\mathbb{R}} \to \mathbb{R}$ is a function defined by $f\big(x\big) = \dfrac{x^2}{1 + x^2}$.

   <u>Note:</u> $\dot{\mathbb{R}}$ is the *effective* and *best* possible *domain* over *set* of *real numbers*.

   The *range* of the *function* is given by the *interval*

   A. $\big(-\infty, 0\big] \cup \big[1, \infty\big)$ B. $\big(0, 1\big]$ C. $\big[0, 1\big)$ D. none of these

5. Let $A = \Big\{a : a \in \mathbb{N}; 1 \leq a \leq 20\Big\}$. Let $B = \Big\{2, 3, 5, 7, 11\Big\}$.

   Let R be a relation given by $R : B \to A = \big\{\big(b, a\big) : b \in B; a \in A;\ b \text{ is the } smallest\ prime\ factor \text{ of } a\big\}$.

   You are allowed to adjust set A or B or both by only removing element(s) not adding any more element(s).

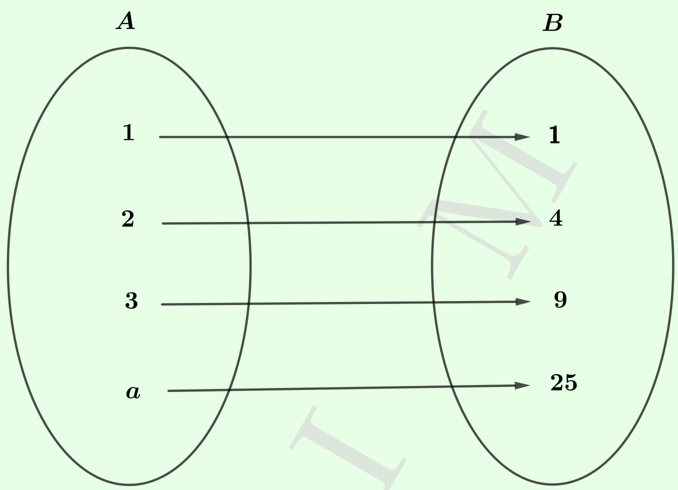
   The defined link is unaltered.

   Then the *smallest number of elements* that can be removed to make this *relation* a *function* is _______.

6. Let $F : \mathbb{R} - \{0\} \to \mathbb{R}$ be a *function* satisfying $2f(x) - 3f\left(\dfrac{1}{x}\right) = x^2$ for *any real* $x \neq 0$.

   Then the value of $4f(2) =$ ___________.

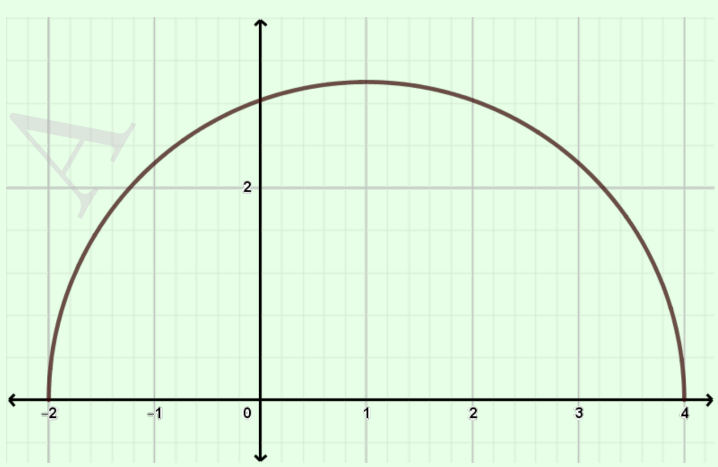
7. $f : A \to B$ is a *function* represented in the *arrow diagram*, as follows:



   Then, the *value* of the *domain element* a

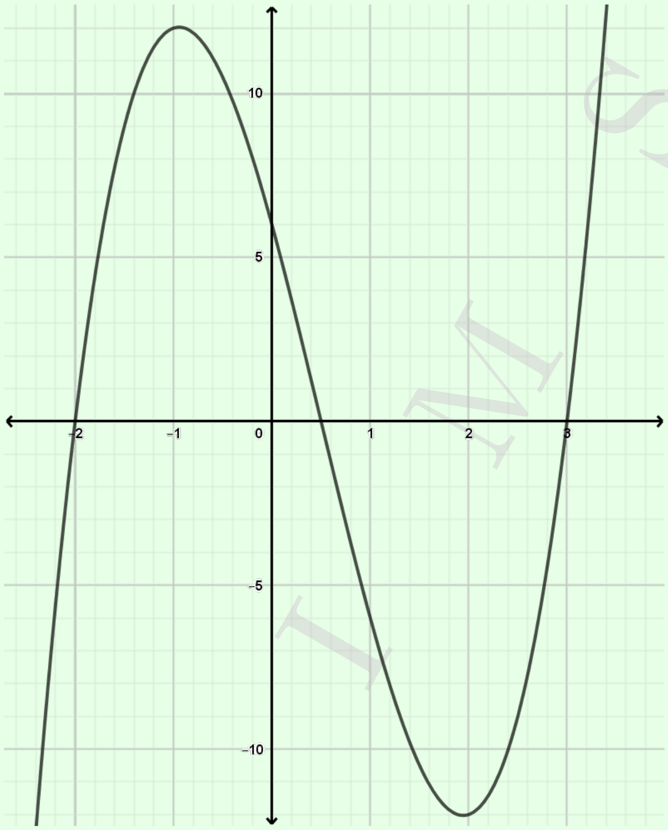
   A. is ± 3 B. is ± 4 C. *cannot be exactly determined* D. ± 5

8. The *function* shown in the following *graph of a semi-circular arc* is $y = \sqrt{\_\_\_ - \left(x - 1\right)^2}$.



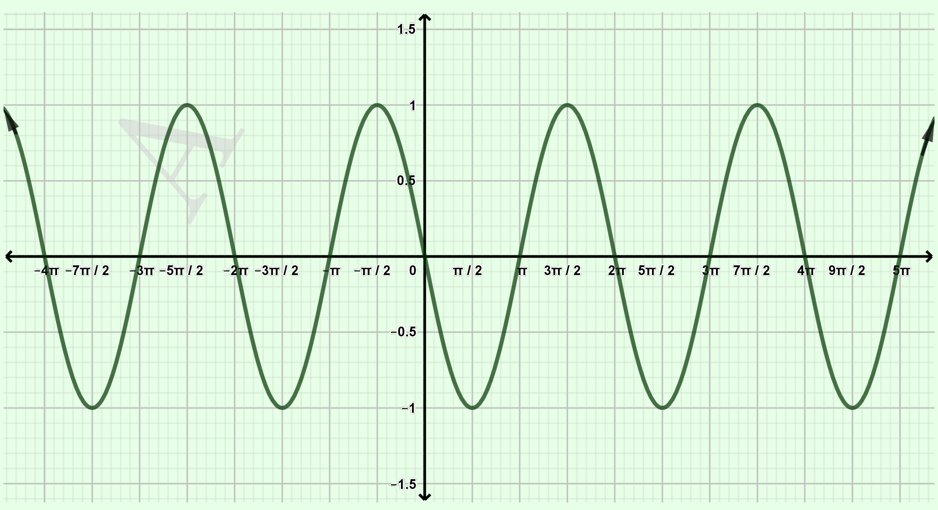
   A. 4 B. 9 C. 16 D. 25

9. The *function* shown in the following *graph* is $y = 2x^3 - 3x^2 - 11x +$_________.



A. 4 B. 5 C. 6 D. 7

10. Which of the following *function* is shown in the *graph*?



A. $y = \cos x$ B. $y = -\cos x$ C. $y = \sin x$ D. $y = -\sin x$

Sadagopan Rajesh COMBO TEST 1 SEPTEMBER 06, 2024

Maximum time: 135 *minutes*

Basics from previous standards, Matrices and Determinants, Relations and Functions
Inverse Trigonometric Functions, Continuity & Differentiability, Application of Derivatives

KEM7 - Advanced Maths for Std 12th only @ *ABIMS*

Try on your own ! Don't use calculators ! Think and Answer !

Name:____ Standard:__________

I Answer the following questions accordingly !

I.I Section - A : <u>Questions on *Concepts*</u>

1. The number of distinct 2×2 matrices that can be formed using all the four elements $0, 1, 1, 2$ is

   A. 6 B. 24 C. 12 D. none of these

2. The number of solutions for matrix A of order 2×2 satisfying $A^2 = A$ is

   A. 0 B. 1 C. 2 D. infinitely many

3. The minor of element -1 in the 1×1 matrix $[-1]$ is

   A. -1 B. 1 C. 0 D. none of these

4. A root of the equation $\begin{vmatrix} 0 & x-a & x-b \\ x+a & 0 & x-c \\ x+b & x+c & 0 \end{vmatrix} = 0$, is

   A. a B. b C. 0 D. 1

5. Let A, B, C be square matrices of same order. Define $A * B = \dfrac{1}{2}(AB + BA)$

   Consider the following statements:

   - $A * I = A$.
   - $A * A = A^2$.
   - $A * B = B * A$.
   - $A * (B - C) = A * B - A * C$.

Then, the number of correct statements is

A. 1 B. 2 C. 3 D. 4

6. $f(x) = \dfrac{x-1}{x+1}$ is a function where $x \in \mathbb{R} - \{-1, 1\}$.

Then $f\left(\dfrac{1}{f(x)}\right)$ equals

A. 1 B. 0 C. x D. $\dfrac{1}{x}$

7. If R is a relation from a finite set A having m elements and a finite set B having m elements, then the number of possible relations from A to B is

A. 2^{mn} B. $2^{mn} - 1$ C. 2^{mn} D. m^n

8. The period of the function $f(x) = \sin\left(\dfrac{2x+3}{6\pi}\right)$ is

A. 6π B. 2π C. $6\pi^2$ D. none of these

9. The domain of the function $f(x) = \dfrac{\sin^{-1}(x-3)}{\sqrt{9-x^2}}$ is

A. $[2, 3]$ B. $[2, 3)$ C. $[1, 2]$ D. $[1, 2)$

10. The value of $\cot^{-1}\left(\cot \dfrac{3\pi}{4}\right)$ is

A. $\dfrac{\pi}{4}$ B. $-\dfrac{\pi}{4}$ C. $\dfrac{3\pi}{4}$ D. $-\dfrac{3\pi}{4}$

11. The interval in which $\sin^{-1} x > \cos^{-1} x$ is

A. $x \in (1, \sqrt{2})$ B. $x \in \left(\dfrac{1}{\sqrt{2}}, 1\right)$ C. $x \in (-1, 1)$ D. $x \in (0, 1)$

12. If $a \sin^{-1} x - b \cos^{-1} x = c$, then $\sin^{-1} x =$

A. 0 B. $\dfrac{b\pi + 2c}{2a + 2b}$ C. $\dfrac{a\pi + 2c}{2a + 2b}$ D. none of these

13. If $x > 0, y > 0$ and $x > y$, then $\tan^{-1}\left(\dfrac{x}{y}\right) + \tan^{-1}\left(\dfrac{x+y}{x-y}\right)$ is equal to

A. $\dfrac{-\pi}{4}$ B. $\dfrac{\pi}{4}$ C. $\dfrac{3\pi}{4}$ D. none of these

14. If $y = |x|$ where $x \in \mathbb{R} - \{0\}$, then $\dfrac{dy}{dx} =$

A. $\dfrac{|x|}{x}$ B. 1 C. -1 D. 0

15. The function defined by

$$f(x) = \begin{cases} x & x < 1 \\ 2-x & 1 \le x \le 2 \\ -2+3x-x^2 & x > 2 \end{cases} \text{ is}$$

A. differentiable at $x = 1$ and $x = 2$

B. not differentiable at $x = 1$ but differentiable at $x = 2$

C. differentiable at $x = 1$

D. not differentiable at $x = 2$

16. The value of k for the function

$$f(x) = \begin{cases} 2x^2+5x+2 & for \;\; x \neq -2 \\ k & for \;\; x = -2 \end{cases}$$ to be continuous at $x = -2$, is

A. 1 B. -1 C. 0 D. 2

17. Given $y = x \tan y$. Then, $\dfrac{dy}{dx}=$

A. $\dfrac{\tan y}{x-x^2-y^2}$ B. $\dfrac{y}{x-x^2-y^2}$ C. $\dfrac{\tan y}{y-x}$ D. $\dfrac{y}{x-x^2-x^2y}$

18. If $x = a\left[\cos\theta + \log_e \tan\left(\dfrac{\theta}{2}\right)\right], y = a\sin\theta$, then $\dfrac{dy}{dx} =$

A. $\cot\theta$ B. $\sec\theta$ C. $\tan\theta$ D. $cosec\ \theta$

19. The equation of normal to the curve $y = x + \sin x \cos x$ at $x = \dfrac{\pi}{2}$

A. $x = \pi$ B. $x = 2$ C. $x + \pi = 0$ D. $x = \dfrac{\pi}{2}$

20. At the origin, the curve $y = x^3 + x$

A. touches the x-axis

B. bisects the angle between the axis

C. makes $60°$ with positive X axis

D. touches the Y-axis

I.II Section - B : Questions on *Applications*

21. If $A = \begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix}, C = \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}$ and $ABC = I$, then the matrix B is

A. $\begin{bmatrix} 7 & 5 \\ -11 & -8 \end{bmatrix}$ B. $\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$ C. $\begin{bmatrix} 7 & 1 \\ 34 & 5 \end{bmatrix}$ D. $\begin{bmatrix} 5 & 3 \\ 13 & 8 \end{bmatrix}$

22. If α, β and γ, λ are the roots of the equations $x^2 - 6x + 3 = 0$ and $y^2 - 4y + 2 = 0$ respectively, then the value of $\begin{vmatrix} \alpha\beta & \gamma\lambda & 1 \\ \alpha+\beta & \gamma+\lambda & 2 \\ \sin(\pi\alpha\beta) & \cos\left(\dfrac{\pi}{2}\gamma\lambda\right) & 1 \end{vmatrix}$ is

A. 0 B. 1 C. 2 D. 6

23. The range of the function $f(x) = 9^x - 3^x + 1$ is

A. $\left[\frac{4}{3}, \infty\right)$ B. $\left[\frac{3}{4}, \infty\right)$ C. $\left(-\infty, \frac{3}{4}\right]$ D. $[0, 1]$

24. Let $f : \mathbb{R} - \left\{\frac{-4}{3}\right\} \longrightarrow \mathbb{R}$ be a function defined as $f(x) = \frac{4x}{3x+4}$.The inverse of f is the map g : Range $f \longrightarrow \mathbb{R} - \left\{\frac{-4}{3}\right\}$ given by

A. $g(y) = \frac{3y}{3-4y}$ B. $g(y) = \frac{4y}{4-3y}$ C. $g(y) = \frac{4y}{3-4y}$ D. $g(y) = \frac{3y}{4-3y}$

25. If $\alpha = \tan^{-1}\left(\frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{\sqrt{1+x^2} + \sqrt{1-x^2}}\right)$, then $x^2 =$

A. $\sin\alpha$ B. $\cos 2\alpha$ C. $\cos\alpha$ D. $\sin 2\alpha$

26. The value of $\cos^{-1} x + \cos^{-1}\left(\frac{x}{2} + \frac{1}{2}\sqrt{3-3x^2}\right)$ where $x \in \left[\frac{1}{2}, 1\right]$, is equal to

A. $\frac{\pi}{6}$ B. $\frac{\pi}{3}$ C. π D. 0

27. The value of $\lim_{x \to 0} \frac{1}{x} \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ is

A. -2 B. 2 C. 1 D. 0

28. The number of points at which $\frac{1}{log|x|}$ is discontinuous , is

A. 1 B. 2 C. 3 D. infinite

29. The point on the parabola $y^2 = 4x$ closest to the point $(5, 2)$ is

A. $(1, -2)$ B. $(1, 2)$ C. $(4, 4)$ D. none of these

30. The minimum value of $64 \sec\theta + 27\ cosec\ \theta$ when $\theta \in \left(0, \frac{\pi}{2}\right)$ is

A. 25 B. 100 C. 125 D. 150

I.III Section - C : Questions on *Applications*

31. The smallest value of natural number $'n'$ for which $A^n = 0$(zero matrix) and
$A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$, is _______

32. Let $D = \begin{vmatrix} 2022! & 2024! & \dfrac{2026!}{2023 \times 2024} \\ 2024! & 2026! & \dfrac{2028!}{2025 \times 2026} \\ 2026! & 2028! & \dfrac{2030!}{2027 \times 2028} \end{vmatrix}$ then $\dfrac{D}{2022! \times 2024! \times 2026!} =$ _______

33. If $y = \cos^2(\tan^{-1}(\sin(\cot^{-1} 3)))$, then $1331y^3 - 3630y^2 + 3300y + 7369 =$ _______

34. The number of real solutions of $\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2+x+1} = \dfrac{\pi}{2}$ is

A. 0 B. 1 C. 2 D. infinitely many

35. The maximum value of $\dfrac{2x^2-2x+2}{x^2+x+1}$ for all real values of x , is _______

36. The altitude of a cylinder of the greatest possible volume,which can be inscribed in a sphere of radius $3\sqrt{3}$ unit, is _______ unit.

37. If the period of $f(x) = \cos 4x + \tan^2 x$ is $\dfrac{k\pi}{4}$,then the value of k is _______

38. The range of the function $f(x) = \dfrac{\sec^2 x - \tan x}{\sec^2 x + \tan x}, \dfrac{-\pi}{2} < x < \dfrac{\pi}{2}$, is $[a, b]$,Then, the value of $3a + 6b$ is _______

39. If f(x)=$\dfrac{\sqrt{1+\sin x} - \sqrt{1-\sin x}}{x}$ is continuous at $x = 0$, then $2f(0) =$ _______

40. If $y = 3\cos(\log_e x) + 4sin(\log_e x)$, then $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + ky = 0$, where constant $k =$ ______

APPLICATION OF VECTORS

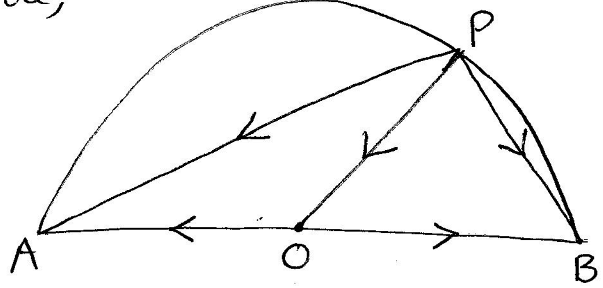
by SADAGOPAN RAJESH

* Vectors can be used to derive certain geometric properties.

Property 1 Angle in a semi-circle is a right angle.

Let AB be the diameter of a circle, whose centre is O.

Let 'P' be a point on the boundary of the semi-circle arc $\overset{\frown}{AB}$ other than A, B, as shown.



Consider $\overrightarrow{PA} \cdot \overrightarrow{PB} = (\overrightarrow{PO} + \overrightarrow{OA}) \cdot (\overrightarrow{PO} + \overrightarrow{OB})$

$$= (\overrightarrow{PO} + \overrightarrow{OA}) \cdot (\overrightarrow{PO} - \overrightarrow{OA})$$

$$= PO^2 - OA^2$$

$$= r^2 - r^2 = 0$$

As $\overrightarrow{PA}$, $\overrightarrow{PB}$ are both non-zero vectors, they must be at right angles to each other.

$\therefore$ $\boxed{\angle APB = 90^\circ.}$ Hence, it is proved.

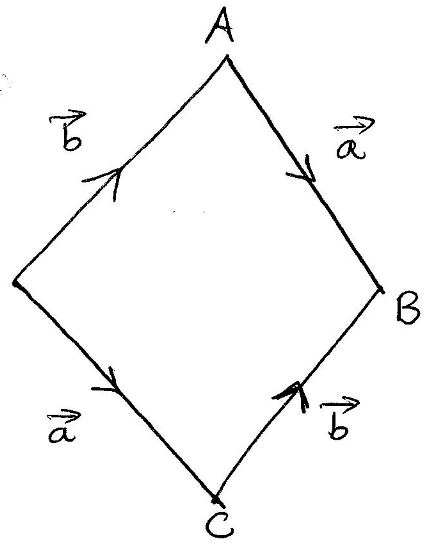
Note that $OP = OA = OB = r$ (the radius of the circle)

$\overrightarrow{OB} = -\overrightarrow{OA}$ as the vectors having magnitude 'r' but have exact opposite directions.

Also, $\overrightarrow{OB}, \overrightarrow{OA}, \overrightarrow{OP}$ are all unequal vectors due to different directions.

Property 2 The diagonals of a Rhombus are at right angles.

Let ABCD be the Rhombus.



Let $\vec{AB} = \vec{DC} = \vec{a}$.

Let $\vec{DA} = \vec{CB} = \vec{b}$.

we have $\vec{AC} = \vec{AD} + \vec{DC}$

$= -\vec{DA} + \vec{DC}$

$\vec{AC} = -\vec{b} + \vec{a}$

(or) $\boxed{\vec{AC} = \vec{a} - \vec{b}}$

Also, $\vec{DB} = \vec{DA} + \vec{AB}$

$= \vec{b} + \vec{a}$

$\therefore \boxed{\vec{DB} = \vec{a} + \vec{b}}$

Now, consider $\vec{AC} \cdot \vec{DB} = (\vec{a} - \vec{b}) \cdot (\vec{a} + \vec{b})$

$= a^2 - b^2 = 0 \quad (\because a = b)$

As $\vec{AC}, \vec{DB}$ are non-zero vectors, they must be at right angles to each other.

$\therefore$ The diagonals AC, DB are at right angles.

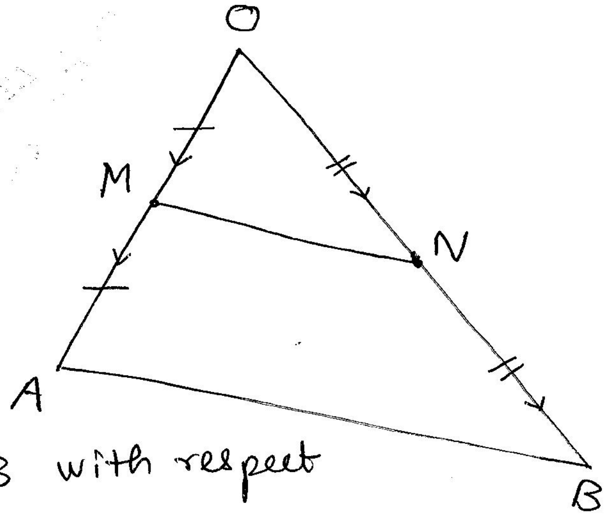
Hence, it is proved.

Note that $AB = BC = CD = DA$ as all the sides of a Rhombus are equal.

$\vec{AB}$, $\vec{CB}$ are unequal vectors as their directions are different.

Theorem 1 MIDPOINT THEOREM

"The line segment joining the midpoints of two sides of a triangle is parallel to the third side and half of it."



Let OAB be the triangle.

Let M, N be the mid points of OA, OB respectively, as shown.

Let the position vectors of A, B with respect to O be given $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$.

As M, N are mid points of OA, OB respectively.

we have $OM = \frac{1}{2} OA$; $ON = \frac{1}{2} OB$.

$\overrightarrow{OM}$ is along $\overrightarrow{OA}$ and $\overrightarrow{ON}$ is along $\overrightarrow{OB}$

$\therefore \overrightarrow{OM} = \frac{1}{2}\overrightarrow{OA} = \frac{\vec{a}}{2}$; $\overrightarrow{ON} = \frac{1}{2}\overrightarrow{OB} = \frac{\vec{b}}{2}$.

$\therefore \overrightarrow{MN} = \overrightarrow{ON} - \overrightarrow{OM} = \frac{\vec{b} - \vec{a}}{2} = \frac{\overrightarrow{OB} - \overrightarrow{OA}}{2} = \frac{\overrightarrow{AB}}{2}$

$\therefore \boxed{\overrightarrow{MN} = \frac{1}{2}\overrightarrow{AB}}$. $\rightarrow$ (1)

This implies that $MN = \frac{1}{2} AB$ and $MN \parallel AB$.

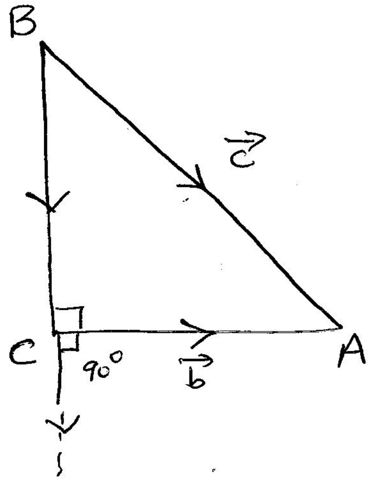
Thus, "MIDPOINT THEOREM" is established using vectors.

Note that (1) gives inference on both magnitude and direction.

Theorem 2

PYTHAGORAS THEOREM

"In a right angled triangle, the square on the hypotenuse equals the sum of the squares of the other two sides."



Consider the right angled triangle ABC.

Let $\vec{BC} = \vec{a}$; $\vec{CA} = \vec{b}$;

$\vec{BA} = \vec{c}$, as indicated

we have $\vec{BC} + \vec{CA} = \vec{BA}$

$\Rightarrow \quad \vec{a} + \vec{b} = \vec{c}$

Squaring on both sides, we get

$$\left|\vec{a} + \vec{b}\right|^2 = \left|\vec{c}\right|^2$$

$\Rightarrow \quad (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) = \vec{c} \cdot \vec{c}$

$\Rightarrow \quad a^2 + b^2 + 2\, \vec{a} \cdot \vec{b} = c^2$

$\Rightarrow \quad a^2 + b^2 + 2\, ab \cos 90^\circ = c^2$ (∵ Angle between $\vec{a}, \vec{b}$ is 90°)

$\Rightarrow \quad \boxed{a^2 + b^2 = c^2}$ as $\cos 90^\circ = 0$.

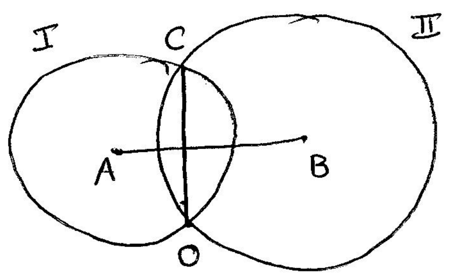
Hence, it is proved.

DOT PRODUCT IS VERY HELPFUL HERE, TO LINK THE SQUARE OF SIDES OF A TRIANGLE.

Property 3 The line segment joining the centres of two intersecting circles is perpendicular to the common chord.

Let A, B be the centres of the intersecting circles.

Let C, O be their points of intersections, as shown.



Let the position vectors of A, B, C with respect to O be given by $\vec{OA} = \vec{a}$; $\vec{OB} = \vec{b}$; $\vec{OC} = \vec{c}$.

we have $\vec{AC} = \vec{OC} - \vec{OA} = \vec{c} - \vec{a}$; $\vec{BC} = \vec{OC} - \vec{OB} = \vec{c} - \vec{b}$

$OA = AC =$ radius of circle I $\Rightarrow |\vec{OA}| = |\vec{AC}|$

$\therefore |\vec{OA}| = |\vec{AC}| \Rightarrow |\vec{OA}|^2 = |\vec{AC}|^2 \Rightarrow |\vec{a}|^2 = |\vec{c} - \vec{a}|^2$

$\Rightarrow \vec{a}\cdot\vec{a} = (\vec{c}-\vec{a})\cdot(\vec{c}-\vec{a}) \Rightarrow \boxed{a^2 = c^2 + a^2 - 2\vec{c}\cdot\vec{a}} \longrightarrow (1)$

Similarly, $OB = BC =$ radius of circle II $\Rightarrow |\vec{OB}| = |\vec{BC}|$

$\therefore |\vec{OB}| = |\vec{BC}| \Rightarrow |\vec{OB}|^2 = |\vec{BC}|^2 \Rightarrow |\vec{b}|^2 = |\vec{c} - \vec{b}|^2$

$\Rightarrow \vec{b}\cdot\vec{b} = (\vec{c}-\vec{b})\cdot(\vec{c}-\vec{b}) \Rightarrow \boxed{b^2 = c^2 + b^2 - 2\vec{c}\cdot\vec{b}} \longrightarrow (2)$

From (1) and (2), we have $c^2 = 2\vec{c}\cdot\vec{a} = 2\vec{c}\cdot\vec{b}$

$\Rightarrow 2\vec{c}\cdot(\vec{b}-\vec{a}) = 0 \Rightarrow \vec{c}\cdot(\vec{b}-\vec{a}) = 0 \Rightarrow \vec{OC}\cdot(\vec{OB}-\vec{OA}) = 0$

$\Rightarrow \vec{OC}\cdot\vec{AB} = 0 \Rightarrow \boxed{OC \perp AB}$ as $\vec{OC}$, $\vec{AB}$ are non-zero vectors.

$\therefore$ OC, the common chord of two circles and AB, the line segment joining the centres of the circles are at right angles to each other. Hence, it is proved.

Property 4 A quadrilateral with one pair of opposite sides equal and parallel, is a parallelogram.

Let ABCD be the quadrilateral where $AB = CD$ and $AB \parallel CD$.

Let the position vectors of A, B, C, D with respect to a reference point 'O' be given by $\vec{OA} = \vec{a}$; $\vec{OB} = \vec{b}$; $\vec{OC} = \vec{c}$; $\vec{OD} = \vec{d}$.

Now, $AB = CD$ and $AB \parallel CD$ $\Rightarrow$ $\boxed{\vec{AB} = \vec{DC}}$.

$\vec{AB} = \vec{DC} \Rightarrow \vec{OB} - \vec{OA} = \vec{OC} - \vec{OD} \Rightarrow \boxed{\vec{b} - \vec{a} = \vec{c} - \vec{d}} \rightarrow (1)$

Adding $(\vec{a} - \vec{c})$ both sides of (1), we get

$\vec{b} - \vec{c} = \vec{a} - \vec{d} \Rightarrow \boxed{\vec{CB} = \vec{DA}} \Rightarrow CB = DA$ and $CB \parallel DA$.

As both the pair of opposite sides of quadrilateral ABCD are respectively parallel to each other, ABCD is a parallelogram.

Hence, it is proved.

Property 5 ABCD is a quadrilateral. P, Q, R, S, the midpoints of sides of AB, BC, CD and DA form the vertices of a parallelogram.

Let the position vectors of A, B, C, D with respect to a reference point 'O' be given by $\vec{OA} = \vec{a}$; $\vec{OB} = \vec{b}$;

$\vec{OC} = \vec{c}$; $\vec{OD} = \vec{d}$.

Then, the position vectors of the midpoints of the sides will be $\vec{OP} = \frac{\vec{OA} + \vec{OB}}{2} = \frac{\vec{a} + \vec{b}}{2}$; $\vec{OQ} = \frac{\vec{OB} + \vec{OC}}{2} = \frac{\vec{b} + \vec{c}}{2}$;

$$\overrightarrow{OR} = \frac{\overrightarrow{OC}+\overrightarrow{OD}}{2} = \frac{\vec{c}+\vec{d}}{2} \; ; \quad \overrightarrow{OS} = \frac{\overrightarrow{OD}+\overrightarrow{OA}}{2} = \frac{\vec{d}+\vec{a}}{2}.$$

Now, $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \frac{\vec{b}+\vec{c}}{2} - \frac{\vec{a}+\vec{b}}{2} = \frac{\vec{c}-\vec{a}}{2}$.

$$\boxed{\overrightarrow{PQ} = \frac{\vec{c}-\vec{a}}{2}} \rightarrow (1)$$

$$\overrightarrow{RS} = \overrightarrow{OS} - \overrightarrow{OR} = \frac{\vec{d}+\vec{a}}{2} - \frac{\vec{c}+\vec{d}}{2} = \frac{\vec{a}-\vec{c}}{2}$$

$$\therefore \overrightarrow{SR} = -\overrightarrow{RS} = \frac{\vec{c}-\vec{a}}{2} \Rightarrow \boxed{\overrightarrow{SR} = \frac{\vec{c}-\vec{a}}{2}} \rightarrow (2)$$

From (1) & (2), we have $\overrightarrow{PQ} = \overrightarrow{SR}$. $\Rightarrow \boxed{PQ = SR}$; $\boxed{PQ \parallel SR}$

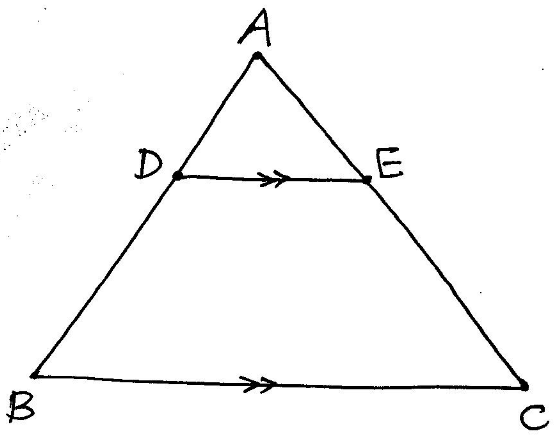
$\therefore$ PQRS is a quadrilateral in which one pair of opposite sides are parallel and equal.

$\Rightarrow$ $\boxed{\text{PQRS is a parallelogram}}$ (by property 4) Hence, it is proved.

Theorem 3 **THALE'S THEOREM (OR) BASIC PROPORTIONALITY THEOREM**

D, E are points on sides AB, AC of ΔABC such that $DE \parallel BC$.

Then, we have $\frac{AD}{DB} = \frac{AE}{EC}$.



We can prove this theorem, using vectors.

Let the position vectors of B, C with respect to A given by $\overrightarrow{AB} = \vec{b}$ and $\overrightarrow{AC} = \vec{c}$. Then, $\overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB} = \vec{c} - \vec{b}$

$$\boxed{\overrightarrow{BC} = \vec{c} - \vec{b}}$$

$DE \parallel BC \Rightarrow \boxed{\overrightarrow{DE} = \lambda\, \overrightarrow{BC}}$ for some positive scalar λ.

This is because $\overrightarrow{DE}, \overrightarrow{BC}$ have same direction.

$\therefore \boxed{\overrightarrow{DE} = \lambda(\vec{c} - \vec{b})} \longrightarrow (1)$

Now, $\boxed{\overrightarrow{AE} = p. \overrightarrow{AC}}$ for some positive scalar p.

This is because $\overrightarrow{AE}$ is along $\overrightarrow{AC}$. (see figure!)

Also, $\boxed{\overrightarrow{AD} = q. \overrightarrow{AB}}$ for some positive scalar q.

we have $\overrightarrow{DE} = \overrightarrow{AE} - \overrightarrow{AD} = p. \overrightarrow{AC} - q. \overrightarrow{AB}$

$\therefore \boxed{\overrightarrow{DE} = p.\overrightarrow{AC} - q.\overrightarrow{AB}} \longrightarrow (2)$

From (1) and (2), we have

$\lambda(\vec{c} - \vec{b}) = p\vec{c} - q\vec{b}$ as $\overrightarrow{AC} = \vec{c}$; $\overrightarrow{AB} = \vec{b}$

$\Rightarrow \boxed{(\lambda - p)\,\vec{c} = (\lambda - q)\,\vec{b}} \longrightarrow (3)$

$\Rightarrow \lambda - p = \lambda - q = 0$ making the equation (3) as $\vec{0} = \vec{0}$,

which is valid.

otherwise $\vec{c}$ is scalar times $\vec{b} \Rightarrow \vec{c}, \vec{b}$ are parallel

vectors, which is not so.

$\therefore \boxed{\lambda = p = q}$ $\therefore \boxed{\overrightarrow{AE} = \lambda. \overrightarrow{AC} \; ; \; \overrightarrow{AD} = \lambda\, \overrightarrow{AB}}$

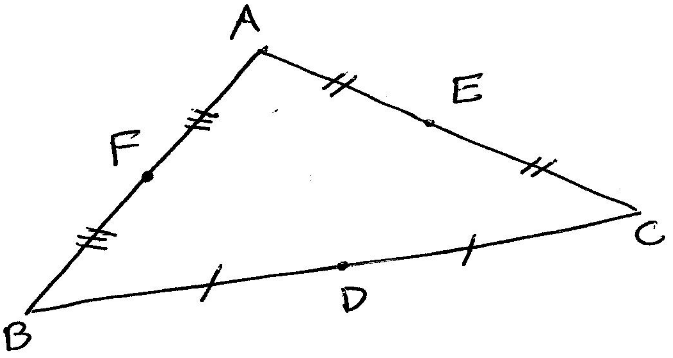
$\Rightarrow AE = \lambda. AC \; ; \; AD = \lambda. AB \Rightarrow \boxed{\dfrac{AE}{AC} = \dfrac{AD}{AB} = \lambda}$

Thus, Thale's Theorem is established using vectors.

Property 6 The medians of a triangle are concurrent. The point of concurrency called as centroid divides each median length in the ratio of $2:1$ internally.

Let A, B, C be the vertices of a triangle whose position vectors are given by $\vec{OA} = \vec{a}$; $\vec{OB} = \vec{b}$; $\vec{OC} = \vec{c}$

Let D, E, F be the midpoints of sides BC, AC, AB respectively.



By section formula, we have

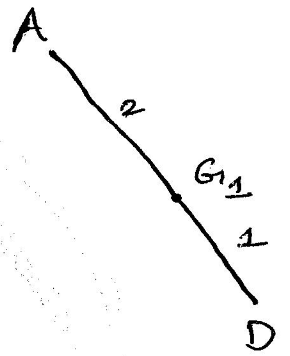
$$\vec{OD} = \frac{\vec{OB} + \vec{OC}}{2} \Rightarrow \boxed{\vec{d} = \frac{\vec{b}+\vec{c}}{2}}$$

$$\vec{OE} = \frac{\vec{OC} + \vec{OA}}{2} \Rightarrow \boxed{\vec{e} = \frac{\vec{c}+\vec{a}}{2}}$$

$$\vec{OF} = \frac{\vec{OA} + \vec{OB}}{2} \Rightarrow \boxed{\vec{f} = \frac{\vec{a}+\vec{b}}{2}}$$

note: O is a reference point in space suitably, other than the points used here.

Let G_1, G_2, G_3 be interior points on medians AD, BE, CF respectively dividing it in the ratio of $2:1$.



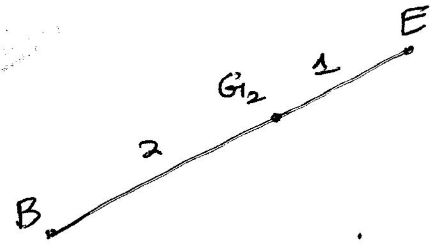
By section formula, we have

$$\vec{OG_1} = \frac{\vec{OA} + 2\,\vec{OD}}{1+2} = \frac{\vec{a} + 2\vec{d}}{3}$$

$$\vec{OG_1} = \frac{\vec{a} + 2\left(\frac{\vec{b}+\vec{c}}{2}\right)}{3} = \frac{\vec{a}+\vec{b}+\vec{c}}{3}$$

$$\therefore \boxed{\vec{OG_1} = \frac{\vec{a}+\vec{b}+\vec{c}}{3}} \longrightarrow (1)$$

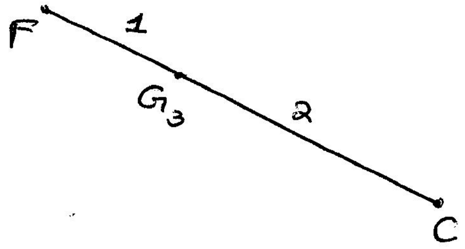
By section formula, we have



$$\overrightarrow{OG_2} = \frac{\overrightarrow{OB} + 2\overrightarrow{OE}}{1+2}$$

$$\overrightarrow{OG_2} = \frac{\vec{b} + 2\left(\frac{\vec{c}+\vec{a}}{2}\right)}{3}$$

$$\boxed{\overrightarrow{OG_2} = \frac{\vec{a}+\vec{b}+\vec{c}}{3}} \longrightarrow (2)$$



Also, $\overrightarrow{OG_3} = \frac{\overrightarrow{OC} + 2\overrightarrow{OF}}{1+2} = \frac{\vec{c} + 2\vec{f}}{3}$

$$\overrightarrow{OG_3} = \frac{\vec{c} + 2\left(\frac{\vec{a}+\vec{b}}{2}\right)}{1+2} = \frac{\vec{a}+\vec{b}+\vec{c}}{3}$$

$$\boxed{\overrightarrow{OG_3} = \frac{\vec{a}+\vec{b}+\vec{c}}{3}} \longrightarrow (3)$$

From (1), (2) and (3), we have $\boxed{\overrightarrow{OG_1} = \overrightarrow{OG_2} = \overrightarrow{OG_3}}$.

All the directed line segments originating from O are equal in magnitude and have same direction.

$\therefore$ G_1, G_2, G_3 are one and the same location, say G.

$\therefore$ G lies on all the three medians AD, BE, CF.

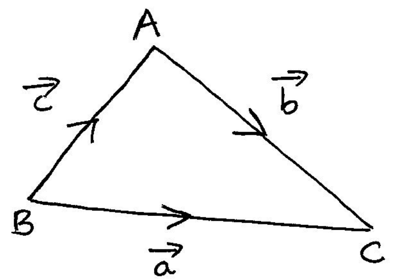
Conversely, we can say that the three medians AD, BE, CF concur at 'G', the centroid of the triangle.

Also, we have $\frac{AG}{GD} = \frac{BG}{GE} = \frac{CG}{GF} = \frac{2}{1}$.

Hence, it is proved.

$\boxed{\text{Rule 1}}$ Cosine rule: In ΔABC, $BC = a$; $CA = b$; $AB = c$. We have

$$a^2 = b^2 + c^2 - 2bc\cos A \;;\; b^2 = c^2 + a^2 - 2ca\cos B \;;\; c^2 = a^2 + b^2 - 2ab\cos C$$



Let $\overrightarrow{AC} = \vec{b}$; $\overrightarrow{BC} = \vec{a}$; $\overrightarrow{BA} = \vec{c}$. Angle between $\vec{c}, \vec{a}$ is B.

Angle between $\vec{b}, \vec{a}$ is C. Angle between $\vec{c}, \vec{b}$ is $(180° - A)$.

We have

$$\boxed{\vec{c} + \vec{b} = \vec{a}}$$

$$|\vec{a}|^2 = |\vec{b} + \vec{c}|^2$$

$$\Rightarrow \vec{a}.\vec{a} = (\vec{b}+\vec{c}).(\vec{b}+\vec{c})$$

$$\Rightarrow a^2 = b^2 + c^2 + 2\,\vec{b}.\vec{c}$$

$$\Rightarrow a^2 = b^2 + c^2 + 2bc\cos(180° - A)$$

$$\Rightarrow \boxed{a^2 = b^2 + c^2 - 2bc\cos A}$$

$$\boxed{\vec{c} = \vec{a} - \vec{b}}$$

$$|\vec{c}|^2 = |\vec{a} - \vec{b}|^2$$

$$\Rightarrow \vec{c}.\vec{c} = (\vec{a} - \vec{b}).(\vec{a} - \vec{b})$$

$$\Rightarrow c^2 = a^2 + b^2 - 2\,\vec{a}.\vec{b}$$

$$\Rightarrow \boxed{c^2 = a^2 + b^2 - 2ab\cos C}$$

$$\boxed{\vec{b} = \vec{a} - \vec{c}}$$

$$|\vec{b}|^2 = |\vec{a} - \vec{c}|^2$$

$$\Rightarrow \vec{b}.\vec{b} = (\vec{a} - \vec{c}).(\vec{a} - \vec{c})$$

$$\Rightarrow b^2 = a^2 + c^2 - 2\,\vec{a}.\vec{c}$$

$$\Rightarrow \boxed{b^2 = a^2 + c^2 - 2ac\cos B}$$

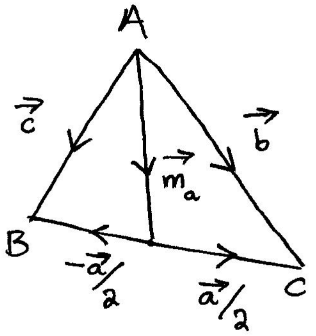
In general, the square of any side length of a triangle equals the sum of the squares of the other two side lengths diminished by twice the product of the other two side lengths and cosine of the angle (vertex angle) opposite to it.

THEOREM 4

APPOLLONIUS THEOREM:

The square of twice the median of a triangle equals twice the sum of the squares of two sides containing the median diminished by ~~twice~~ the square of the side on which the median falls.

Proof:



$$\vec{m_a} - \vec{a}/2 = \vec{c}$$

$$\vec{m_a} + \vec{a}/2 = \vec{b}$$

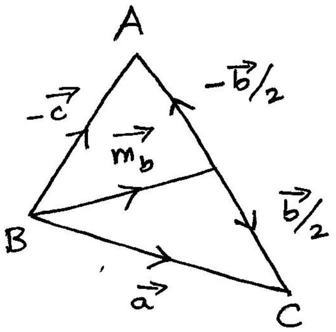
Squaring and adding, we get

$$2\left(m_a^2 + \frac{a^2}{4}\right) = c^2 + b^2$$

$$\frac{4m_a^2 + a^2}{4} = \frac{c^2+b^2}{2}$$

$$4m_a^2 + a^2 = 2(c^2+b^2)$$

$$\boxed{(2m_a)^2 = 2(c^2+b^2) - a^2}$$



$$\vec{m_b} - \frac{\vec{b}}{2} = -\vec{c}$$

$$\vec{m_b} + \frac{\vec{b}}{2} = \vec{a}$$

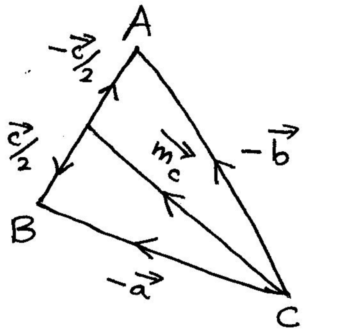
Squaring and adding, we get

$$2\left(m_b^2 + \frac{b^2}{4}\right) = a^2 + c^2$$

$$\frac{4m_b^2 + b^2}{4} = \frac{a^2+c^2}{2}$$

$$4m_b^2 + b^2 = (a^2+c^2)\times 2$$

$$\boxed{(2m_b)^2 = 2(a^2+c^2) - b^2}$$



$$\vec{m_c} - \vec{c}/2 = -\vec{b}$$

$$\vec{m_c} + \vec{c}/2 = -\vec{a}$$

Squaring and adding, we get

$$2\left(m_c^2 + \frac{c^2}{4}\right) = a^2 + b^2$$

$$\frac{4m_c^2 + c^2}{4} = \frac{a^2+b^2}{2}$$

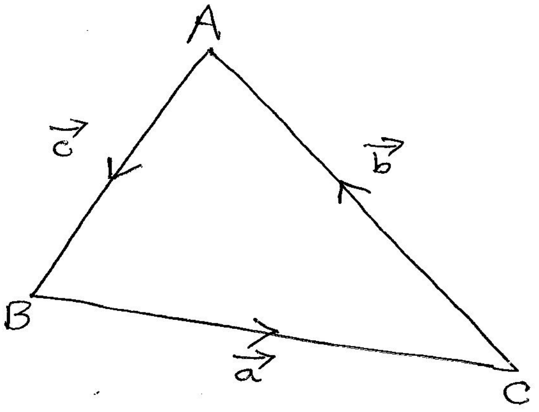
$$4m_c^2 + c^2 = 2(a^2+b^2)$$

$$\boxed{(2m_c)^2 = 2(a^2+b^2) - c^2}$$

Thus, the relation between each median with the sides of a triangle is established in "Appolonius theorem".

Rule 2 <u>Sine rule</u> : In ΔABC, $BC = a$; $CA = b$; $AB = c$.

We have $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$ (or) $a : b : c = \sin A : \sin B : \sin C$.



Let $\vec{AB} = \vec{c}$; $\vec{BC} = \vec{a}$; $\vec{CA} = \vec{b}$.

We have $\boxed{\vec{a} + \vec{b} + \vec{c} = \vec{0}} \rightarrow (1)$

<u>Pre-applying</u> the cross product of $\vec{a}$ on both sides of (1), we get

$$\vec{a} \times \vec{a} + \vec{a} \times \vec{b} + \vec{a} \times \vec{c} = \vec{a} \times \vec{0}$$

$$\Rightarrow \vec{0} + \vec{a} \times \vec{b} + \vec{a} \times \vec{c} = \vec{0} \Rightarrow \vec{a} \times \vec{b} - \vec{c} \times \vec{a} = \vec{0}$$

$$\Rightarrow \boxed{\vec{a} \times \vec{b} = \vec{c} \times \vec{a}} \longrightarrow (2)$$

Pre-applying the cross product of $\vec{c}$ on both sides of (1), we get

$$\vec{c} \times \vec{a} + \vec{c} \times \vec{b} + \vec{c} \times \vec{c} = \vec{c} \times \vec{0}$$

$$\Rightarrow \vec{c} \times \vec{a} + \vec{c} \times \vec{b} + \vec{0} = \vec{0} \Rightarrow \vec{c} \times \vec{a} - \vec{b} \times \vec{c} = \vec{0}$$

$$\Rightarrow \boxed{\vec{c} \times \vec{a} = \vec{b} \times \vec{c}} \longrightarrow (3)$$

From (2) and (3), we get $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$.

As the vectors are equal, their magnitudes are also equal.

$$\therefore |\vec{a} \times \vec{b}| = |\vec{b} \times \vec{c}| = |\vec{c} \times \vec{a}|$$

$$\Rightarrow ab \sin(180^\circ - C) = bc \sin(180^\circ - A) = ca \sin(180^\circ - B)$$

$$\Rightarrow \boxed{ab \sin C = bc \sin A = ca \sin B} \longrightarrow (4)$$

(4) $\div\ abc \Rightarrow \dfrac{\sin C}{c} = \dfrac{\sin A}{a} = \dfrac{\sin B}{b}$. Thus, Sine rule is established.

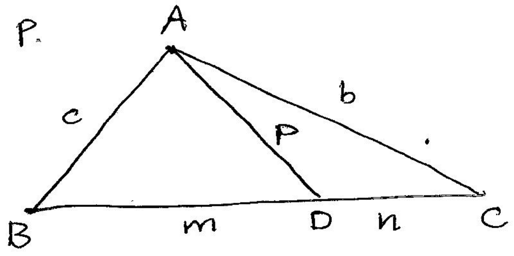
THEOREM 5 — STEWART'S THEOREM :

In ΔABC, AD is the cevian equal to p.

AD divides BC such that

$BD = m$; $CD = n$.

$AB = c$; $AC = b$; $BC = a$.



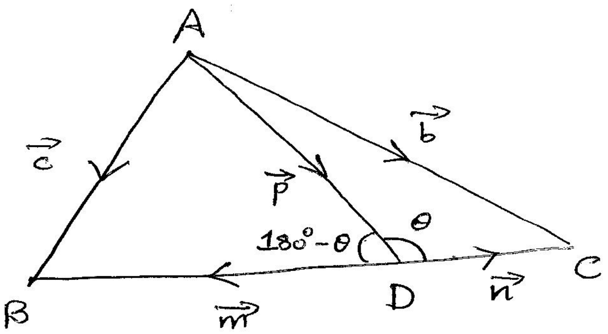
Then, these five segments are connected by the relation

$$\boxed{p^2 + mn = \frac{mb^2 + nc^2}{m+n}}$$

Proof by vectors:

we have $\vec{AD} = \vec{p}$; $\vec{AB} = \vec{c}$;

$\vec{AC} = \vec{b}$; $\vec{DB} = \vec{m}$; $\vec{DC} = \vec{n}$.



we have $\vec{p} + \vec{m} = \vec{c}$

and $\vec{p} + \vec{n} = \vec{b}$

squaring each of the above two equations, we get

$$|\vec{p} + \vec{m}|^2 = |\vec{c}|^2$$

$$(\vec{p} + \vec{m}) \cdot (\vec{p} + \vec{m}) = \vec{c} \cdot \vec{c}$$

$$p^2 + m^2 + 2\vec{p} \cdot \vec{m} = c^2$$

$$p^2 + m^2 + 2pm \cos(180^\circ - \theta) = c^2$$

$$p^2 + m^2 + 2pm(-\cos\theta) = c^2$$

$$\Rightarrow \boxed{np^2 + nm^2 - 2pmn\cos\theta = nc^2} \rightarrow (1)$$

$$|\vec{p} + \vec{n}|^2 = |\vec{b}|^2$$

$$(\vec{p} + \vec{n}) \cdot (\vec{p} + \vec{n}) = \vec{b} \cdot \vec{b}$$

$$p^2 + n^2 + 2\vec{p} \cdot \vec{n} = b^2$$

$$p^2 + n^2 + 2pn\cos\theta = b^2$$

$$p^2 + n^2 + 2pn\cos\theta = b^2$$

$$\Rightarrow \boxed{mp^2 + mn^2 + 2pmn\cos\theta = mb^2} \rightarrow (2)$$

Now, ① + ② $\Rightarrow$

$$np^2 + nm^2 + mp^2 + mn^2 - 2pmn\cos\theta + 2pmn\cos\theta = nc^2 + mb^2$$

$$\Rightarrow (m+n)p^2 + mn(m+n) = mb^2 + nc^2$$

$$\Rightarrow (m+n)(p^2 + mn) = mb^2 + nc^2$$

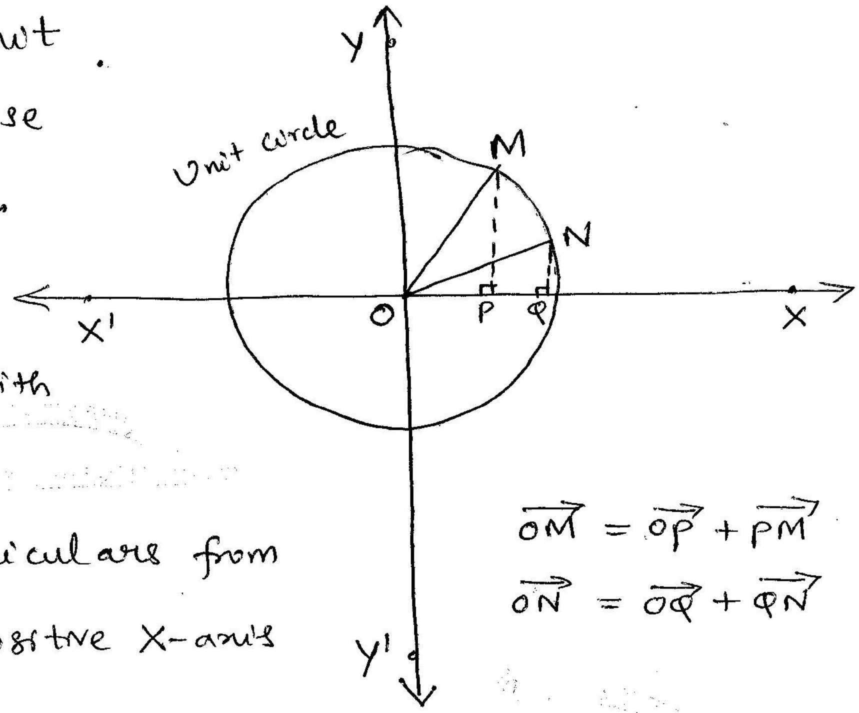
$$\Rightarrow \boxed{p^2 + mn = \frac{mb^2 + nc^2}{m+n}}$$

. Hence, it is proved.

Property 7 $\quad \cos(A-B) = \cos A \cdot \cos B + \sin A \cdot \sin B$

$\quad \cos(A+B) = \cos A \cdot \cos B - \sin A \cdot \sin B$

Let M, N be point on unit circle of X-Y plane, whose centre is origin, as shown.



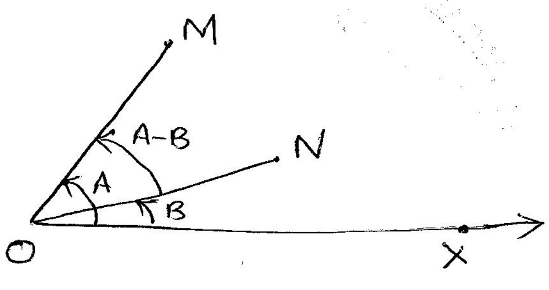
Let OM, ON makes angles A, B respectively with positive X-axis.

$\vec{OM} = \vec{OP} + \vec{PM}$

$\vec{ON} = \vec{OQ} + \vec{QN}$

Let the foot of perpendiculars from M, N fall at P, Q on positive X-axis respectively.

We have $\vec{OM} \cdot \vec{ON} = OM \times ON \times \cos(A-B)$



$$\Rightarrow (\cos A\,\vec{i} + \sin A\,\vec{j}) \cdot (\cos B\,\vec{i} + \sin B\,\vec{j})$$

$$= 1 \times 1 \times \cos(A-B)$$

$$\Rightarrow \boxed{\cos A \cos B + \sin A \sin B = \cos(A-B)}$$

Let N' be the reflection of N about X-axis.

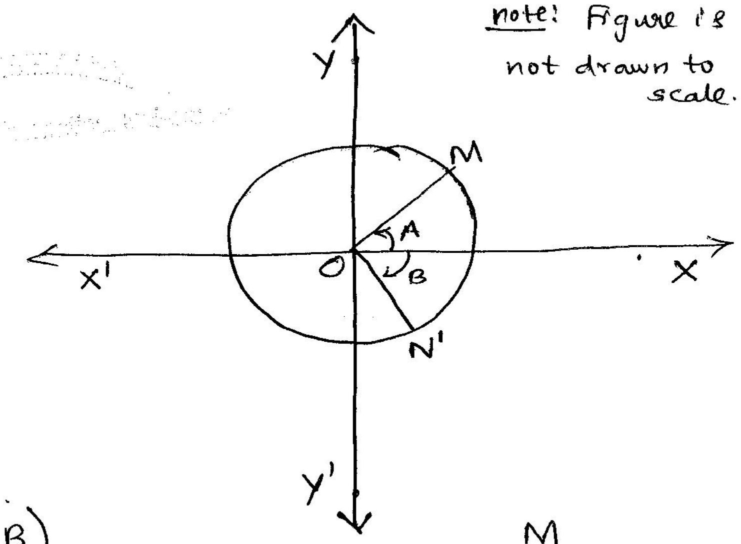
$\overrightarrow{OM} = \overrightarrow{OP} + \overrightarrow{PM}$

$\overrightarrow{OM} = \cos A\,\vec{i} + \sin A\,\vec{j}$

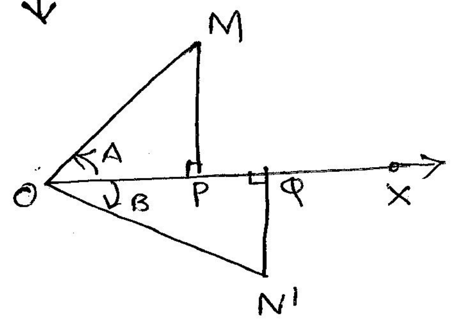
$\overrightarrow{ON'} = \overrightarrow{OQ} + \overrightarrow{QN'}$

$\overrightarrow{ON'} = \cos B\,\vec{i} + \sin B\,(-\vec{j})$

$\overrightarrow{ON'} = \cos B\,\vec{i} - \sin B\,\vec{j}$



$\overrightarrow{OM} \cdot \overrightarrow{ON'} = OM \times ON' \times \cos(A+B)$



$\Rightarrow (\cos A\,\vec{i} + \sin A\,\vec{j}) \cdot (\cos B\,\vec{i} - \sin B\,\vec{j})$

$= 1 \times 1 \times \cos(A+B)$

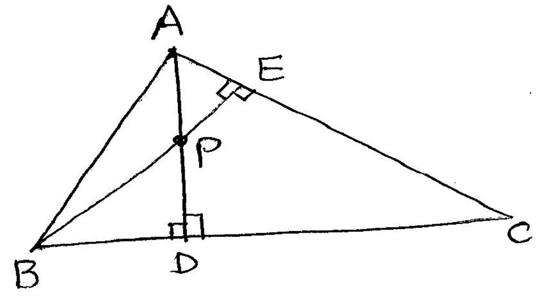
$\Rightarrow \boxed{\cos A \cdot \cos B - \sin A \cdot \sin B = \cos(A+B)}$

Note: $\overrightarrow{OP}, \overrightarrow{OQ}$ are in the direction of positive X-axis.

$\overrightarrow{PM}$ is in the direction of positive Y-axis.

$\overrightarrow{QN'}$ is in the direction of negative Y-axis.

Property 8 The altitudes of a triangle are concurrent.



Let the two altitudes AD, BE of ΔABC intersect at point P, as shown.

Let the position vectors of A, B, C, P be $\vec{a}, \vec{b}, \vec{c}$ and $\vec{p}$ with reference to a point 'O'.

$\therefore AP \perp BC$ and $BP \perp AC \Rightarrow \overrightarrow{AP} \cdot \overrightarrow{BC} = \overrightarrow{BP} \cdot \overrightarrow{AC} = 0$.

$\therefore (\vec{p} - \vec{a}) \cdot (\vec{c} - \vec{b}) = 0$ and $(\vec{p} - \vec{b}) \cdot (\vec{c} - \vec{a}) = 0$

$\Rightarrow \boxed{\vec{p} \cdot (\vec{c} - \vec{b}) = \vec{a} \cdot \vec{c} - \vec{a} \cdot \vec{b}} \rightarrow (1)$; $\boxed{\vec{p} \cdot (\vec{c} - \vec{a}) = \vec{b} \cdot \vec{c} - \vec{a} \cdot \vec{b}} \rightarrow (2)$

$$① - ② \Rightarrow \vec{p}\cdot(\cancel{\vec{c}} - \vec{b} - \cancel{\vec{c}} + \vec{a}) = \vec{a}\cdot\vec{c} - \cancel{\vec{a}\cdot\vec{b}} - \vec{b}\cdot\vec{c} + \cancel{\vec{a}\cdot\vec{b}}$$

$$\Rightarrow \vec{p}\cdot(\vec{a}-\vec{b}) = (\vec{a}-\vec{b})\cdot\vec{c}$$

$$\Rightarrow \vec{p}\cdot(\vec{a}-\vec{b}) = \vec{c}\cdot(\vec{a}-\vec{b})$$ $\because$ vector dot product is commutative.

$$\Rightarrow (\vec{p}-\vec{c})\cdot(\vec{a}-\vec{b}) = 0$$

$$\Rightarrow (\overrightarrow{OP}-\overrightarrow{OC})\cdot(\overrightarrow{OA}-\overrightarrow{OB}) = 0$$

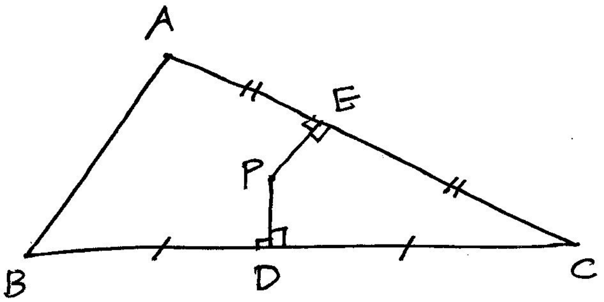
$$\Rightarrow \overrightarrow{CP}\cdot\overrightarrow{BA} = 0 \Rightarrow \boxed{CP \perp AB.}$$

$\therefore$ The third altitude from vertex C to side AB passes through point 'P'.

$\therefore$ All the three altitudes of a triangle are concurrent.

Property 9 The perpendicular bisectors of a triangle are concurrent.

Let the two perpendicular bisectors of sides BC and AC intersect at P, as shown.



Let the position vectors of A, B, C, D, E, P with reference to a point 'O', be given by $\vec{a}, \vec{b}, \vec{c}, \vec{d}, \vec{e}, \vec{p}$ respectively.

That is, $\overrightarrow{OA} = \vec{a}$; $\overrightarrow{OB} = \vec{b}$; $\overrightarrow{OC} = \vec{c}$; $\overrightarrow{OD} = \vec{d}$; $\overrightarrow{OE} = \vec{e}$; $\overrightarrow{OP} = \vec{p}$.

D, E are midpoints of sides BC and AC respectively.

$$\overrightarrow{OD} = \frac{\overrightarrow{OB}+\overrightarrow{OC}}{2} = \frac{\vec{b}+\vec{c}}{2} \quad ; \quad \overrightarrow{OE} = \frac{\overrightarrow{OC}+\overrightarrow{OA}}{2} = \frac{\vec{c}+\vec{a}}{2}.$$

$DP \perp BC$ and $EP \perp CA$.

$\therefore \overrightarrow{DP} \cdot \overrightarrow{BC} = 0$ and $\overrightarrow{EP} \cdot \overrightarrow{CA} = 0$

$(\overrightarrow{OP} - \overrightarrow{OD}) \cdot (\overrightarrow{OC} - \overrightarrow{OB}) = 0$ and $(\overrightarrow{OP} - \overrightarrow{OE}) \cdot (\overrightarrow{OA} - \overrightarrow{OC}) = 0$

$(\vec{p} - \vec{d}) \cdot (\vec{c} - \vec{b}) = 0$ and $(\vec{p} - \vec{e}) \cdot (\vec{a} - \vec{c}) = 0$

$\left(\vec{p} - \dfrac{\vec{c} + \vec{b}}{2}\right) \cdot (\vec{c} - \vec{b}) = 0$ and $\left(\vec{p} - \dfrac{\vec{c} + \vec{a}}{2}\right) \cdot (\vec{a} - \vec{c}) = 0$

$(2\vec{p} - (\vec{c} + \vec{b})) \cdot (\vec{c} - \vec{b}) = 0$ and $(2\vec{p} - (\vec{c} + \vec{a})) \cdot (\vec{a} - \vec{c}) = 0$

$2\vec{p} \cdot (\vec{c} - \vec{b}) = (\vec{c} + \vec{b}) \cdot (\vec{c} - \vec{b})$; $2\vec{p} \cdot (\vec{a} - \vec{c}) = (\vec{a} + \vec{c}) \cdot (\vec{a} - \vec{c})$

$\Rightarrow 2\vec{p} \cdot (\vec{c} - \vec{b}) = c^2 - b^2$; $2\vec{p} \cdot (\vec{a} - \vec{c}) = a^2 - c^2$

$\Rightarrow 2\vec{p} \cdot (\cancel{\vec{c}} - \vec{b} + \vec{a} - \cancel{\vec{c}}) = \cancel{c^2} - b^2 + a^2 - \cancel{c^2}$

$\Rightarrow 2\vec{p} \cdot (\vec{a} - \vec{b}) = a^2 - b^2$ (or) $2\vec{p} \cdot (\vec{a} - \vec{b}) = (\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b})$

$\Rightarrow (2\vec{p} - (\vec{a} + \vec{b})) \cdot (\vec{a} - \vec{b}) = 0$

$\Rightarrow \left(\vec{p} - \dfrac{\vec{a} + \vec{b}}{2}\right) \cdot (\vec{a} - \vec{b}) = 0$

$\Rightarrow (\overrightarrow{OP} - \overrightarrow{OF}) \cdot (\overrightarrow{OA} - \overrightarrow{OB}) = 0$ where F is mid point of AB.

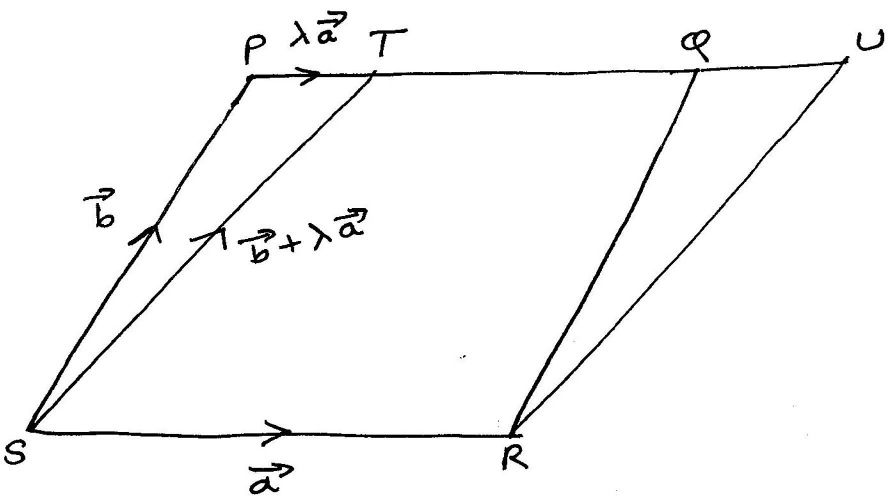
$\Rightarrow \overrightarrow{FP} \cdot \overrightarrow{BA} = 0$ or $\boxed{FP \perp BA}$

$\therefore$ The third perpendicular bisector of side AB also passes through 'P'.

$\therefore$ The perpendicular bisectors of a triangle are concurrent.

Property 10 The parallelogram upon the same base lying between the same parallel lines are equal in area.

Let PQRS and RSTU be the two parallelograms having the common base RS, as shown.



Let $\vec{SR} = \vec{a}$; $\vec{SP} = \vec{b}$;

$\vec{PT} = \lambda\vec{a}$ for some scalar 'λ'.

This is because PT (part of PQ) is parallel to SR.

Now, Area of parallelogram PQRS $= |\vec{a} \times \vec{b}|$, in vector form.

$$\text{Area of parallelogram RSTU} = |\vec{SR} \times \vec{ST}|$$

$$= |\vec{a} \times (\vec{b} + \lambda\vec{a})|$$

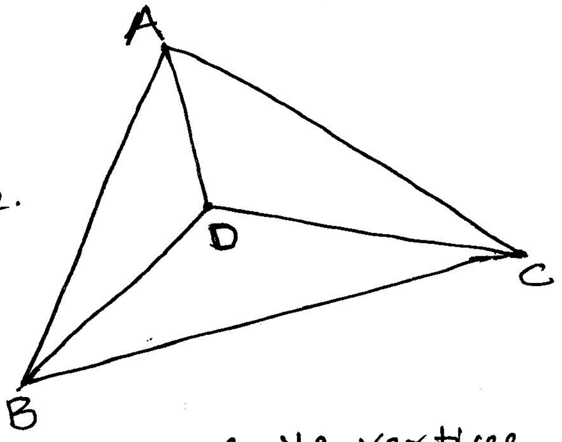
$$= |\vec{a} \times \vec{b} + \lambda\,\vec{a} \times \vec{a}|$$

$$= |\vec{a} \times \vec{b}| \quad \because \vec{a} \times \vec{a} = \vec{0}$$

$\therefore$ Area of parallelograms having a common base lying between the same parallel lines are equal.

* Note that the area of parallelogram in vector form is derived from the geometric result of Area = base × height. More than proving the property, one can think the above as a verification by vector method!

Property 11 ABCD is a tetrahedron. Let G_1, G_2, G_3, G_4 be the centroids of $\triangle BCD$, $\triangle CDA$, $\triangle DAB$, $\triangle ABC$ respectively. Then, the line segments AG_1, BG_2, CG_3, DG_4 are concurrent.

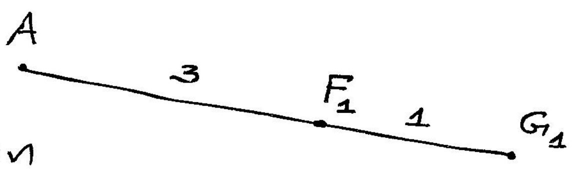


Let the position vectors of A, B, C, D be $\vec{OA} = \vec{a}$, $\vec{OB} = \vec{b}$, $\vec{OC} = \vec{c}$, $\vec{OD} = \vec{d}$ where 'O' is a reference point in space.

We know that the position vector of the centroid of a triangle is equal to the average of the position vectors of its vertices.

Triangle	It's centroid	Position vector of the centroid
$\triangle BDC$	G_1	$\vec{OG_1} = \dfrac{\vec{b}+\vec{c}+\vec{d}}{3}$
$\triangle CDA$	G_2	$\vec{OG_2} = \dfrac{\vec{c}+\vec{d}+\vec{a}}{3}$
$\triangle DAB$	G_3	$\vec{OG_3} = \dfrac{\vec{d}+\vec{a}+\vec{b}}{3}$
$\triangle ABC$	G_4	$\vec{OG_4} = \dfrac{\vec{a}+\vec{b}+\vec{c}}{3}$

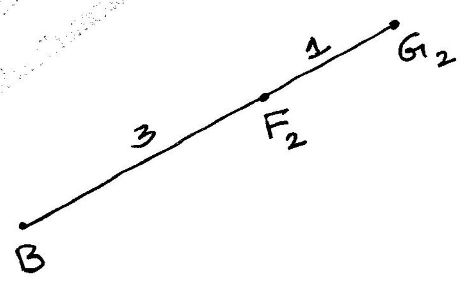
Now, consider the line segment AG_1.



Let F_1 divide AG_1 internally in the ratio of $3:1$

By section formula, we have $\vec{OF_1} = \dfrac{3 \cdot \vec{OG_1} + 1 \cdot \vec{OA}}{3+1}$

$$\overrightarrow{OF_1} = \frac{3\cdot\left(\frac{\vec{b}+\vec{c}+\vec{d}}{3}\right) + \vec{a}}{3+1} \Rightarrow \boxed{\overrightarrow{OF_1} = \frac{\vec{a}+\vec{b}+\vec{c}+\vec{d}}{4}} \rightarrow (1)$$



Now, consider the line segment BG_2.

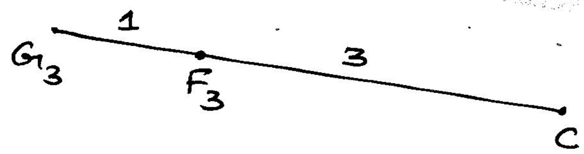
Let F_2 divide BG_2 internally in the ratio of $3:1$.

By section formula, we have $\overrightarrow{OF_2} = \dfrac{3.\overrightarrow{OG_2} + 1.\overrightarrow{OB}}{3+1}$

$$\therefore \overrightarrow{OF_2} = \frac{3\cdot\left(\frac{\vec{c}+\vec{d}+\vec{a}}{3}\right) + \vec{b}}{3+1} \Rightarrow \boxed{\overrightarrow{OF_2} = \frac{\vec{a}+\vec{b}+\vec{c}+\vec{d}}{4}} \rightarrow (2)$$

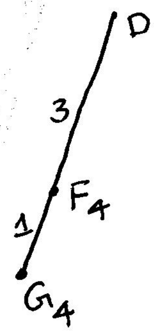
Now, consider CG_3.

Let F_3 divide CG_3 internally in the ratio of $3:1$.



By section formula, we have $\overrightarrow{OF_3} = \dfrac{3.\overrightarrow{OG_3} + 1.\overrightarrow{OC}}{3+1}$

$$\therefore \overrightarrow{OF_3} = \frac{3.\left(\frac{\vec{d}+\vec{a}+\vec{b}}{3}\right) + \vec{c}}{3+1} \Rightarrow \boxed{\overrightarrow{OF_3} = \frac{\vec{a}+\vec{b}+\vec{c}+\vec{d}}{4}} \rightarrow (3)$$



Now, consider the line segment DG_4.

Let F_4 divide DG_4 internally in the ratio of $3:1$.

By section formula, we have $\overrightarrow{OF_4} = \dfrac{3.\overrightarrow{OG_4} + 1.\overrightarrow{OD}}{3+1}$

$$\therefore \overrightarrow{OF_4} = \frac{3.\left(\frac{\vec{a}+\vec{b}+\vec{c}}{3}\right) + \vec{d}}{3+1} \Rightarrow \boxed{\overrightarrow{OF_4} = \frac{\vec{a}+\vec{b}+\vec{c}+\vec{d}}{4}} \rightarrow (4)$$

From (1), (2), (3) and (4), we have $\overrightarrow{OF_1} = \overrightarrow{OF_2} = \overrightarrow{OF_3} = \overrightarrow{OF_4}$.

All the directed line segments originating from O to points F_1, F_2, F_3 and F_4 are equal in magnitude and have same direction.

$\therefore$ F_1, F_2, F_3, F_4 are one and the same location, say F.

$\therefore$ F lies on all the four line segments AG_1, BG_2, CG_3 and DG_4.

Conversely, we can say that these four line segments are concurrent.

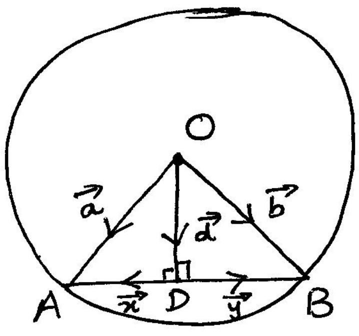
Also, we have $\dfrac{AF}{FG_1} = \dfrac{BF}{FG_2} = \dfrac{CF}{FG_3} = \dfrac{DF}{FG_4} = \dfrac{3}{1}$.

Hence, it is proved.

$\boxed{\text{Property 12}}$ The perpendicular drawn from the centre of a circle to a chord bisects it.

Let the centre of the circle be O.

Let the chord be AB and D is the foot of the perpendicular from O to it.



Let the position vectors of A, B, D be given by $\overrightarrow{OA} = \vec{a}$; $\overrightarrow{OB} = \vec{b}$; $\overrightarrow{OD} = \vec{d}$.

Let $\overrightarrow{DA} = \vec{x}$ and $\overrightarrow{DB} = \vec{y}$

$OD \perp AB \Rightarrow \overrightarrow{DA} \cdot \overrightarrow{OD} = 0$ and $\overrightarrow{OD} \cdot \overrightarrow{DB} = 0$

$OA = OB =$ radius of the circle.

$\therefore \; |\vec{OA}| = |\vec{OB}| \Rightarrow |\vec{OA}|^2 = |\vec{OB}|^2 \Rightarrow |\vec{a}|^2 = |\vec{b}|^2$

$\Rightarrow |\vec{d} + \vec{x}|^2 = |\vec{d} + \vec{y}|^2$

$\Rightarrow d^2 + x^2 + 2\,\vec{d}\cdot\vec{x} = d^2 + y^2 + 2\,\vec{d}\cdot\vec{y}$

$\Rightarrow x^2 = y^2 \qquad \because \vec{d}\cdot\vec{x} = \vec{d}\cdot\vec{y} = 0$

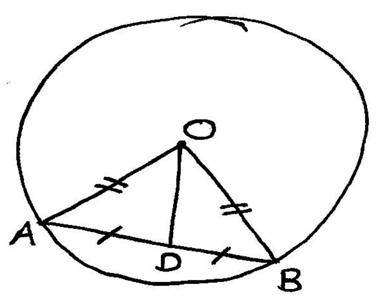
$\Rightarrow \boxed{x = y}$ as $x, y > 0$

$\therefore$ The perpendicular drawn from the centre of a circle bisects the chord.

Property 13 The line segment joining the midpoint of a chord and the centre of a circle is perpendicular to it.

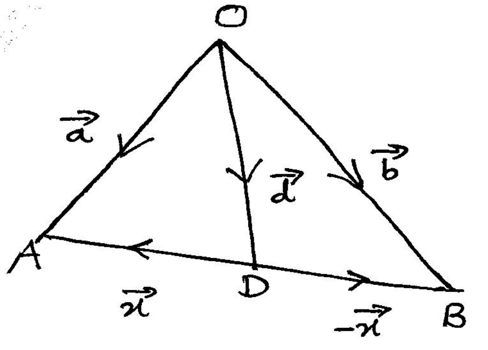
Let the centre of the circle be 'O'.

Let AB be the chord of the circle whose mid point is 'D'.



Let the position vectors of A, D, B with respect to O be given by

$\vec{OA} = \vec{a}$; $\vec{OB} = \vec{b}$; $\vec{OD} = \vec{d}$.



Let $\vec{DA} = \vec{x} \Rightarrow \vec{DB} = -\vec{x}$.

$\vec{DA}, \vec{DB}$ are of same magnitude but have opposite directions.

we have $\vec{OA} = \vec{OD} + \vec{DA}$ and $\vec{OB} = \vec{OD} + \vec{DB}$.

$\therefore$ $\vec{a} = \vec{d} + \vec{x}$ and $\vec{b} = \vec{d} + \vec{y} = \vec{d} - \vec{x}$.

$\Rightarrow$ $|\vec{a}|^2 = |\vec{b}|^2$ $\because$ $a = b =$ radius of the circle

$\Rightarrow$ $|\vec{d} + \vec{x}|^2 = |\vec{d} - \vec{x}|^2$

$\Rightarrow$ $d^2 + x^2 + 2\,\vec{d}\cdot\vec{x} = d^2 + x^2 - 2\,\vec{d}\cdot\vec{x}$

$\Rightarrow$ $4\,\vec{d}\cdot\vec{x} = 0$ $\Rightarrow$ $\vec{d}\cdot\vec{x} = 0$ $\Rightarrow$ $OD \perp DA$

(or) $\boxed{OD \perp AB}$.

$\therefore$ The line segment joining the midpoint of a chord and the centre of a circle is perpendicular to it.

Hence, it is proved.

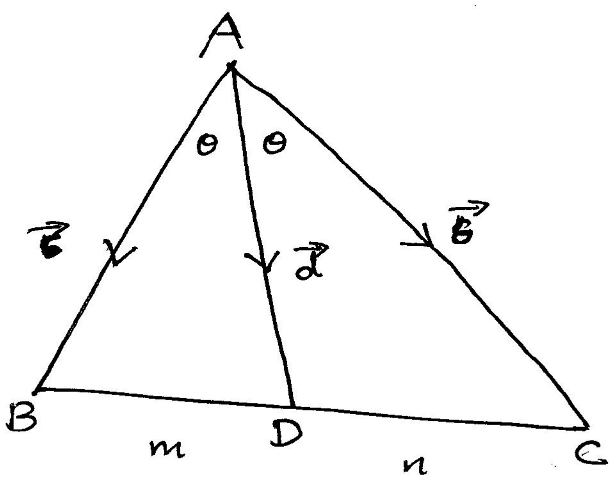
THEOREM 6 Angle bisector theorem :

In ΔABC, AD is the internal angle bisector of $\angle BAC$.

Then $\dfrac{BD}{DC} = \dfrac{AB}{AC}$.

Let the position vectors of B, D, C with reference to point A be given by $\vec{AB} = \vec{c}$; $\vec{AC} = \vec{b}$;

$\vec{AD} = \vec{d}$. Let $\dfrac{BD}{DC} = \dfrac{m}{n}$.



AD bisects $\angle BAC = 2\theta$

$\therefore$ $\angle BAD = \angle CAD = \theta$

By section formula, we have

$$\overrightarrow{AD} = \frac{m\,\overrightarrow{AC} + n\,\overrightarrow{AB}}{m+n} = \frac{m\vec{b} + n\vec{c}}{m+n}$$

$$\Rightarrow \boxed{\vec{d} = \frac{m\vec{b} + n\vec{c}}{m+n}} \longrightarrow (1)$$

Now, consider $cbd\cos\theta = cbd\cos\theta$

$$\Rightarrow c(bd\cos\theta) = b(cd\cos\theta)$$

$$\Rightarrow c(\vec{b}\cdot\vec{d}) = b(\vec{c}\cdot\vec{d})$$

$$\Rightarrow c\left(\vec{b}\cdot\frac{m\vec{b}+n\vec{c}}{m+n}\right) = b\left(\vec{c}\cdot\frac{m\vec{b}+n\vec{c}}{m+n}\right)$$

$$\Rightarrow c\left(\vec{b}\cdot(m\vec{b}+n\vec{c})\right) = b\left(\vec{c}\cdot(m\vec{b}+n\vec{c})\right)$$

$$\Rightarrow c(mb^2 + n\,\vec{b}\cdot\vec{c}) = b(m\,\vec{b}\cdot\vec{c} + nc^2)$$

$$\Rightarrow mcb^2 - nc^2b + (nc - mb)(\vec{b}\cdot\vec{c}) = 0$$

$$\Rightarrow bc(mb - nc) + (nc - mb)\,bc\cos 2\theta = 0$$

$$\Rightarrow (mb - nc)\cdot bc(1 - \cos 2\theta) = 0$$

$$\Rightarrow mb - nc = 0 \quad \because b, c > 0 \text{ and } \cos 2\theta \neq 1$$

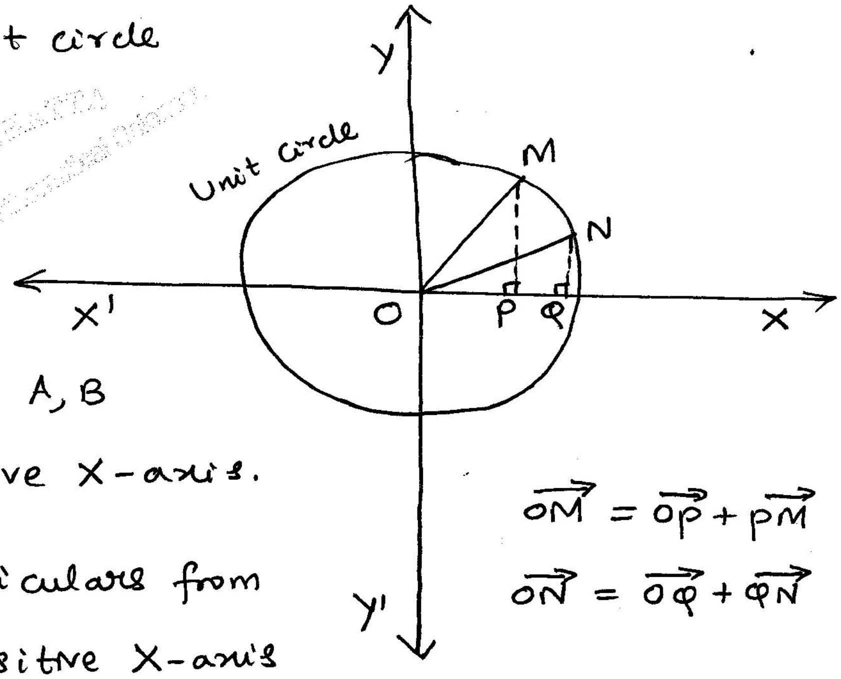
$$\Rightarrow \frac{m}{n} = \frac{c}{b} \quad \therefore \boxed{\frac{BD}{DC} = \frac{c}{b} = \frac{AB}{AC}.}$$

Hence, it is proved.

Property 14 $\sin(A-B) = \sin A \cdot \cos B - \cos A \cdot \sin B$

$\sin(A+B) = \sin A \cdot \cos B + \cos A \cdot \sin B$

Let M, N be point on unit circle of X-Y plane, whose centre is origin, as shown.

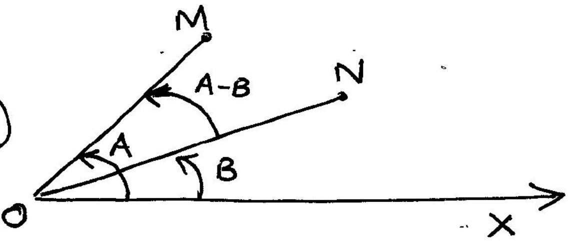


$\vec{OM} = \vec{OP} + \vec{PM}$

$\vec{ON} = \vec{OQ} + \vec{QN}$

Let OM, ON makes angles A, B respectively with positive X-axis.

Let the foot of perpendiculars from M, N fall at P, Q on positive X-axis respectively.



We have $\vec{OM} \times \vec{ON} = OM.ON.\sin(A-B)(-\hat{k})$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos A & \sin A & 0 \\ \cos B & \sin B & 0 \end{vmatrix} = 1 \times 1 \times \sin(A-B) \times -\vec{k}$$

This is because $\vec{OM} = \cos A\,\vec{i} + \sin A\,\vec{j}$; $\vec{ON} = \cos B\,\vec{i} + \sin B\,\vec{j}$

$$(\cos A \sin B - \sin A \cos B)\,\vec{k} = \sin(A-B) \times -\vec{k}$$

$$\Rightarrow \cos A \sin B - \sin A \cos B = -\sin(A-B)$$

$$\Rightarrow \boxed{\sin(A-B) = \sin A \cos B - \cos A \sin B}$$

Hence, it is proved.

Let N' be the reflection of N about X-axis.

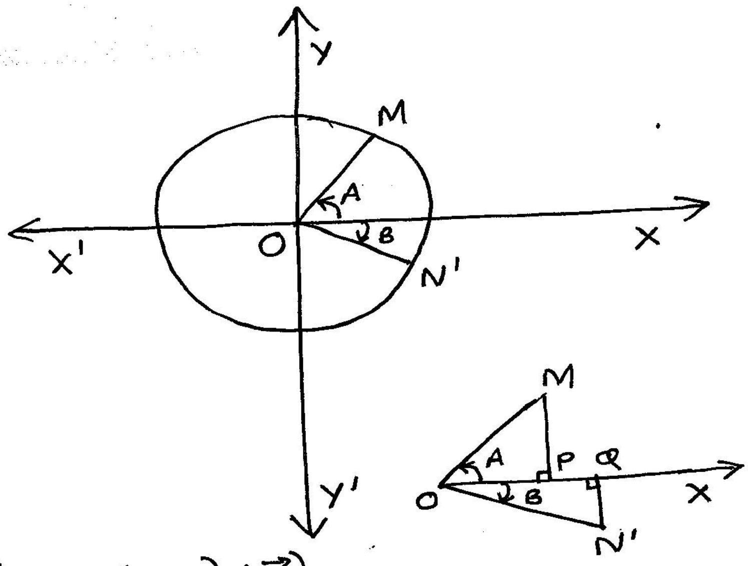
$$\vec{OM} = \vec{OP} + \vec{PM}$$

$$\vec{OM} = \cos A\, \vec{i} + \sin A\, \vec{j}$$

$$\vec{ON'} = \vec{OQ} + \vec{QN'}$$

$$\vec{ON'} = \cos B\, \vec{i} + \sin B\, (-\vec{j})$$

$$\vec{ON'} = \cos B\, \vec{i} - \sin B\, \vec{j}$$



Consider $\vec{OM} \times \vec{ON'} = OM \times ON' \times \sin(A+B)\,(-\vec{k})$

$$\Rightarrow \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos A & \sin A & 0 \\ \cos B & -\sin B & 0 \end{vmatrix} = 1 \times 1 \times \sin(A+B) \times -\vec{k}$$

$$\Rightarrow \vec{k}\,(-\cos A \sin B - \sin A \cos B) = \sin(A+B) \times -\vec{k}$$

$$\Rightarrow \boxed{\sin(A+B) = \sin A \cdot \cos B + \cos A \cdot \sin B}$$

Note: $\vec{OP}$, $\vec{OQ}$ are in the direction of positive X-axis.

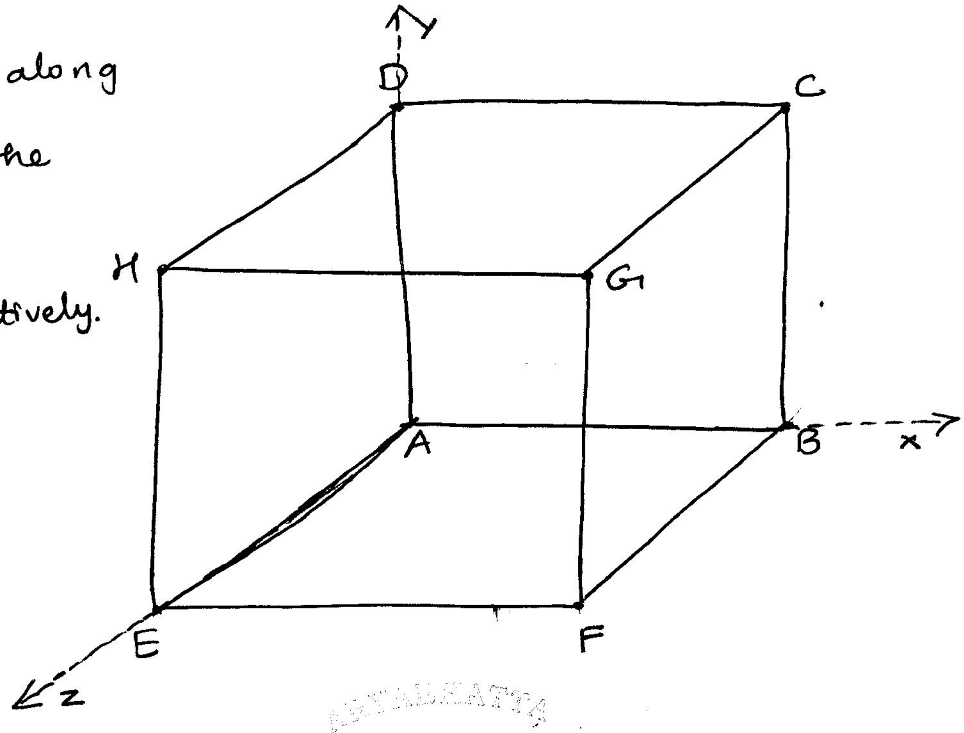
$\vec{PM}$ is in the direction of positive Y-axis.

$\vec{QN'}$ is in the direction of negative Y-axis.

Property 15 The Angle between the diagonals of a cube is $\cos^{-1}(1/3)$.

Let ABCDEFGH be the cube.

Let AB, AD, AE lie along the positive X-axis, the positive Y-axis and positive Z-axis respectively.



Let A be the origin (or) reference point.

Let the cube be a unit cube, each edge of length 1 unit.

$\therefore$ The co-ordinates of A, B, C, D, E, F, G, H are respectively given by $A(0,0,0)$, $B(1,0,0)$, $C(1,1,0)$, $D(0,1,0)$, $E(0,0,1)$, $F(1,0,1)$, $G(1,1,1)$, $H(0,1,1)$.

The diagonals of the cube are the line segments that connects the opposite corners.

They are AG, BH, CE and DF.

We have $\overrightarrow{AG} = \vec{i} + \vec{j} + \vec{k}$.

$$\overrightarrow{BH} = \overrightarrow{AH} - \overrightarrow{AB} = (\vec{j} + \vec{k}) - (\vec{i}) = \vec{j} + \vec{k} - \vec{i}$$

$$\overrightarrow{CE} = \overrightarrow{AE} - \overrightarrow{AC} = (\vec{k}) - (\vec{i} + \vec{j}) = \vec{k} - \vec{i} - \vec{j}$$

$$\overrightarrow{DF} = \overrightarrow{AF} - \overrightarrow{AD} = (\vec{i} + \vec{k}) - \vec{j} = \vec{i} + \vec{k} - \vec{j}$$

Let θ be the angle between the diagonals AG and BH.

We have $\overrightarrow{AG} \cdot \overrightarrow{BH} = AG \times BH \times \cos\theta$

$$(\vec{i} + \vec{j} + \vec{k}) \cdot (-\vec{i} + \vec{j} + \vec{k}) = \sqrt{1^2+1^2+1^2}\ \sqrt{1^2+1^2+1^2} \cdot \cos\theta$$

$\Rightarrow (1\times -1) + (1\times 1) + (1\times 1) = \sqrt{3}.\sqrt{3}.\cos\theta$

$\Rightarrow -1+1+1 = 3.\cos\theta \Rightarrow \cos\theta = 1/3$

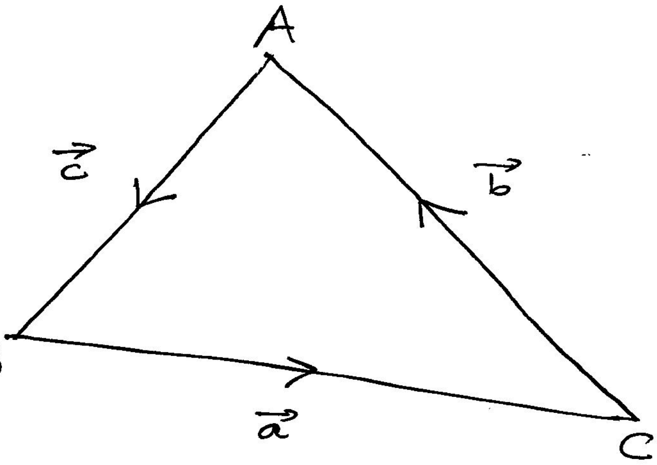
$\Rightarrow \boxed{\theta = \cos^{-1}(1/3)}$

It can be verified that the angle between any pair of the four diagonals of the cube is $\cos^{-1}(1/3)$.

Rule 3 Projection rule :

In ΔABC, $a = b\cos C + c\cos B$; $b = c\cos A + a\cos C$
$c = a\cos B + b\cos A$ where $a = BC$; $b = CA$; $c = AB$.

Let $\vec{AB} = \vec{c}$; $\vec{BC} = \vec{a}$;
$\vec{CA} = \vec{b}$, as shown.



we have

$\boxed{\vec{a}+\vec{b}+\vec{c} = \vec{0}} \rightarrow (1)$

Applying dot product of $\vec{a}$ on both sides of (1), we get $\vec{a}.\vec{a} + \vec{a}.\vec{b} + \vec{a}.\vec{c} = \vec{a}.\vec{0}$

$a^2 + ab\cos(180^\circ - C) + ac\cos(180^\circ - B) = 0$

$\Rightarrow a^2 - ab\cos C - ac\cos B = 0$

$\Rightarrow a^2 = ab\cos C + ac\cos B$

$\Rightarrow \boxed{a = b\cos C + c\cos B} \rightarrow (2)$ since $a > 0$.

Applying dot product of $\vec{b}$ on both sides of ①, we get

$$\vec{b}\cdot\vec{a} + \vec{b}\cdot\vec{b} + \vec{b}\cdot\vec{c} = \vec{b}\cdot\vec{0}$$

$$\Rightarrow \quad ba\cos(180^\circ - C) + b^2 + bc\cos(180^\circ - A) = 0$$

$$\Rightarrow \quad -ba\cos C + b^2 - bc\cos A = 0$$

$$\Rightarrow \quad b^2 = ba\cos C + bc\cos A$$

$$\Rightarrow \quad \boxed{b = a\cos C + c\cos A} \longrightarrow ③ \quad \text{Since } b > 0.$$

Applying dot product of $\vec{c}$ on both sides of ①, we get

$$\vec{c}\cdot\vec{a} + \vec{c}\cdot\vec{b} + \vec{c}\cdot\vec{c} = \vec{c}\cdot\vec{0}$$

$$\Rightarrow \quad ca\cos(180^\circ - B) + cb\cos(180^\circ - A) + c^2 = 0$$

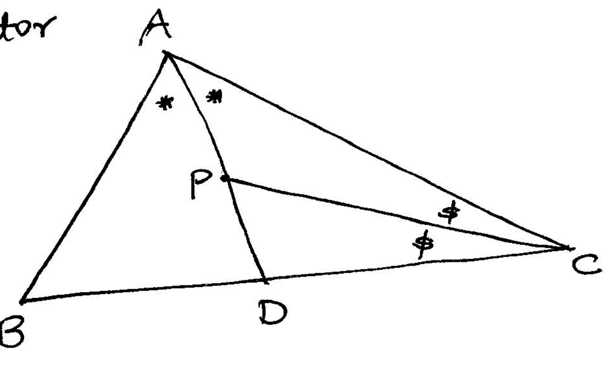
$$\Rightarrow \quad -ca\cos B - cb\cos A + c^2 = 0$$

$$\Rightarrow \quad c^2 = ca\cos B + cb\cos A$$

$$\Rightarrow \quad \boxed{c = a\cos B + b\cos A} \longrightarrow ④ \quad \text{Since } c > 0.$$

Property 16 The angle bisectors of a triangle are concurrent.

Let AD be the internal angle bisector of $\angle BAC$. Let the internal angle bisector of $\angle ACB$ intersects AD at point P, as shown.



Let $AB = c$; $BC = a$; $CA = b$.

Let the position vectors of points A, B, C, D, P be given by $\vec{OA} = \vec{a}$; $\vec{OB} = \vec{b}$; $\vec{OC} = \vec{c}$; $\vec{OD} = \vec{d}$; $\vec{OP} = \vec{p}$ where O is a reference point in space.

By Angle bisector theorem, we have

$$\frac{BD}{DC} = \frac{AB}{AC} \Rightarrow \boxed{\frac{BD}{DC} = \frac{c}{b}} \longrightarrow (1)$$

Let $BD = ct$; $DC = bt$ for some scalar t.

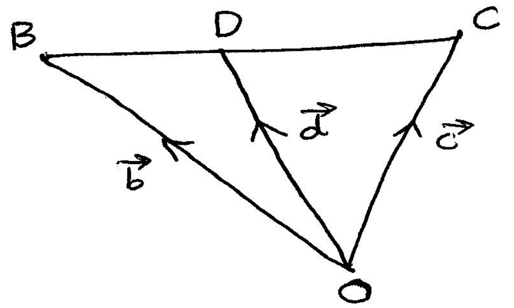
$$BC = BD + DC \Rightarrow a = (c+b)\,t \Rightarrow \boxed{t = \frac{a}{c+b}} \rightarrow (2)$$

$$\therefore BD = ct = \frac{ac}{c+b} \Rightarrow \boxed{BD = \frac{ac}{c+b}} \longrightarrow (3)$$

$$DC = bt = \frac{ab}{c+b} \Rightarrow \boxed{DC = \frac{ab}{c+b}} \longrightarrow (4)$$

By section formula, we have

$$\boxed{\vec{d} = \frac{b\vec{b} + c\vec{c}}{b+c}} \rightarrow (5)$$



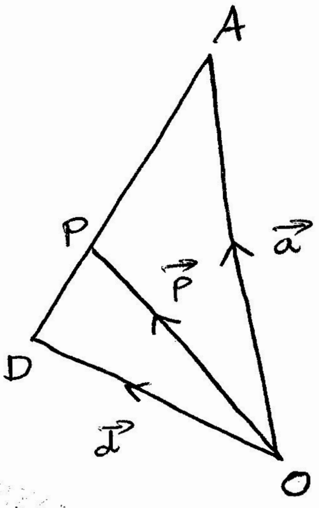
This is because $\frac{BD}{DC} = \frac{c}{b}$.

By Angle bisector theorem, we have

$$\frac{AP}{PD} = \frac{AC}{DC} \Rightarrow \frac{AP}{PD} = \frac{b}{(ab/c+b)} = \frac{b+c}{a}$$

$$\therefore \boxed{\frac{AP}{PD} = \frac{b+c}{a}} \longrightarrow (6)$$

By section formula, we have



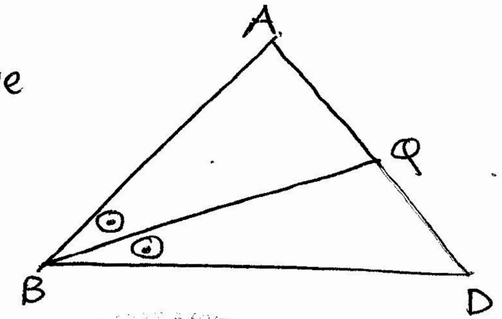
$$\vec{p} = \frac{(b+c)\,\vec{d} + a\,\vec{a}}{b+c+a}$$

$$\Rightarrow \vec{p} = \frac{(b+c)\dfrac{b\vec{b}+c\vec{c}}{b+c} + a\vec{a}}{a+b+c}$$

$$\Rightarrow \boxed{\vec{p} = \frac{a\vec{a} + b\vec{b} + c\vec{c}}{a+b+c}} \longrightarrow (7)$$

Let the angle bisector of $\angle ABC$ meet AD at Q.

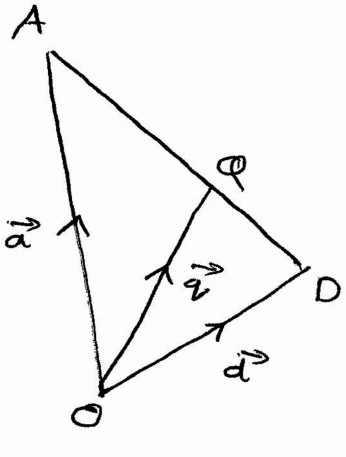
By Angle bisector theorem, we have



$$\frac{AB}{BD} = \frac{AQ}{QD}$$

$$\Rightarrow \frac{AQ}{QD} = \frac{c}{\left(ac/_{c+b}\right)} = \frac{b+c}{a}$$

By section formula, we have



$$\overrightarrow{OQ} = \vec{q} = \frac{(b+c)\,\vec{d} + a\vec{a}}{b+c+a}$$

$$\Rightarrow \boxed{\overrightarrow{OQ} = \frac{a\vec{a} + b\vec{b} + c\vec{c}}{a+b+c}} \rightarrow (8)$$

From (7) & (8), we have $\overrightarrow{OP} = \overrightarrow{OQ}$ $\Rightarrow$ P, Q are the same points.

$\therefore$ The angle bisector of B cuts AD at point P.

$\therefore$ The internal angle bisectors of a triangle are concurrent.